

# Algebraic observer design for a class of uniformly-observable nonlinear systems: Application to 2-link robotic manipulator

Salim Ibrir, *Member, IEEE*

**Abstract**—We propose a globally convergent observer for three-state nonlinear systems verifying the uniform complete observability condition. By constructing a time-varying differentiator, we are then able to reproduce the first and the second derivatives of the system output without imposing the boundedness of the states or the output. By exploiting the algebraic observability of the system, we show that the unmeasured states can be reproduced as nonlinear outputs of the time-varying differentiator. This new technique has several advantages over classical observer design methodologies that are basically related to the form of the system nonlinearities. It will be shown that the complete uniform observability implies the existence of globally convergent observer without major restriction on the system nonlinearities. Illustrative example is provided to demonstrate the efficiency of the proposed design.

**Index Terms**—Nonlinear Systems; Time-varying Systems; Exact differentiation; Observer Design; System Theory.

## I. INTRODUCTION

NONLINEAR observer design for dynamical systems has been the subject of many research papers where several approaches have been used to reconstruct the unmeasured states from input and output measurements. As it has been reported in the literature, the complexity of state estimation depends on the nature of the system nonlinearities, the kind of the input applied to the system being observed, and the form of the system output that plays a key role in the linearization of the error dynamics and its stability. The non availability of a straightforward design method for constructing an observer for a given nonlinear system has created many challenging methods of observation that are generally dependent upon state transformations, the structure of the system being observed, the form of nonlinearities, the boundedness of the system states or the Lipschitz property of the system nonlinearities. Among the most popular strategies that have been employed to build an observer, we cite error-linearization-based algorithms [1], [2], [3], [4], Lyapunov design procedures [5], [6] and sliding-mode observer design [7], [8]. Other challenging procedures as numerical observation methods [9], neural-network observation techniques [10], algebraic nonlinear observer design [11], and circle-criterion observation methods [12], [13], [14], [15]. In case where the system fails to be put in certain form of observability, high-gain observer design reveals as a robust technique that is often used to reconstruct the system states under the assumption that the vector nonlinearity is globally or locally

Lipschitz, see [16], [17], [18], [19], [20], [21]. However, the Lipschitz constraint is not always verified and prevents generally the global convergence of the high-gain observer. Moreover, the existence of the observer gain is conditioned by the value of the Lipschitz constant which is generally required to be small enough, see [21] for more details. Even though the circle-criterion observer design is conceptually free from the information of the Lipschitz constant [22], [14], this interesting design remains limited to systems with positive-gradient nonlinearities.

In this paper, we show that the uniform algebraic observability implies the existence of a nonlinear observer for three-state dynamical systems. The nonlinear observer is constructed as a time-varying differentiator with nonlinear outputs. These nonlinear outputs will serve as tools to reconstruct the states of the system being observed. we show that the estimates converge with the same rate of convergence of the differentiator states whatever the form of the nonlinearities. The proposed technique can be generalized for larger class of systems that have not necessarily triangular structures as it is shown through an example of a working 2-link robot.

## II. PRELIMINARIES

For the clarity of the next developments, we would rather recall some concepts of nonlinear observability.

### A. Definitions

*Definition 1:* Consider the nonlinear system

$$\begin{cases} \dot{x} = f(x, u), \\ y = h(x), \end{cases} \quad (1)$$

where  $x = x(t) \in \mathcal{M} \subset \mathbb{R}^n$  represents the system state vector,  $f(\cdot, \cdot)$  is smooth vector with  $f(0, 0) = 0$  and  $u(t) \in \mathcal{U} \subset \mathbb{R}^m$  is the control input. The output nonlinearity  $y = y(t) = h(x(t)) \in \mathbb{R}^p$  is supposed to be smooth with  $h(0) = 0$ . We say that system (1) is observable if for every two different initial conditions  $x_0$  and  $\bar{x}_0$  there exist an interval  $[0, T]$ ,  $T \in \mathbb{R}_{>0}$  and an admissible control  $u(t)$  defined on  $[0, T]$  such that the associated outputs  $y(x_0, u(t))$ ,  $y(\bar{x}_0, u(t))$  are not identically equal on  $[0, T]$ . We say, in this case, that the control input  $u(t)$  distinguishes the pair  $(x_0, \bar{x}_0)$  on  $[0, T]$ .

*Definition 2:* Consider system (1). The control input  $u(t) \in \mathcal{U} \subset \mathbb{R}^m$  is said universal on  $[0, T]$ , if it distinguishes every different initial states  $(x_0, \bar{x}_0)$  on  $[0, T]$ .

*Definition 3:* System (1) is said uniformly observable if every admissible control  $u(t)$  defined on  $[0, T]$ , is a universal one.

Salim Ibrir is with University of Trinidad and Tobago, Pt. Lisas Campus, P.O. Box. 957, Esperanza Road, Brechin Castle, Couva, Trinidad, W.I., email: salim.ibrir@utt.edu.tt

*Definition 4:* System (1) is said to be algebraically observable if there exist two positive integers  $\mu$  and  $\nu$  such that

$$x(t) = \phi\left(y, \dot{y}, \ddot{y}, \dots, y^{(\mu)}, u, \dot{u}, \ddot{u}, \dots, u^{(\nu)}\right)(t), \quad (2)$$

where  $\phi(\cdot) : \mathbb{R}^{(\mu+1)p} \times \mathbb{R}^{(\nu+1)m} \mapsto \mathbb{R}^n$  is a differentiable vector valued nonlinearity that depends on the inputs, the outputs, and their respective higher derivatives.

The result of the following Lemma will be used in the proof of the main statement of this paper.

*Lemma 1:* Let  $f(t) : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$  be a uniformly bounded and a continuously differentiable function for all  $t \geq 0$ , then

$$i) \lim_{t \rightarrow \infty} e^{-t^2} \int_0^t e^{\tau^2} f(\tau) d\tau = 0, \quad (3)$$

and

$$ii) \lim_{t \rightarrow \infty} e^{-t^2} \int_0^t f(\tau) \left( \int_0^\tau e^{\zeta^2} d\zeta \right) d\tau = 0, \quad (4)$$

*Proof:* See [11].

### B. System description

Consider the three-state nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + g_1(x_1, u), \\ \dot{x}_2 &= x_3 + g_2(x_1, x_2, u), \\ \dot{x}_3 &= g_3(x_1, x_2, x_3, u), \\ y &= x_1, \end{aligned} \quad (5)$$

where  $u = u(t) \in \mathbb{R}^m$  is a  $\mathcal{C}^1$  control input,  $y = y(t) \in \mathbb{R}$  is the system measured output and  $g_1(x_1, u)$ ,  $g_2(x_1, x_2, u)$ , and  $g_3(x_1, x_2, x_3, u)$  are smooth, non-singular and continuous nonlinearities. We assume that the following assumptions hold for  $t \geq 0$ .

*Assumption 1:* For a given input  $u(t)$ , the system output  $y(t)$  is continuously measured, smooth, and twice continuously differentiable with respect to time. The input  $u(t)$  is differentiable and is not necessarily bounded.

*Assumption 2:* For a given input  $u \in \mathbb{R}^m$ , the system states do not leave any compact set. In other words, the system trajectories are well-defined for all  $t \geq 0$  such that for any instant  $t \geq 0$ , we can find a large compact set  $\Omega_t$  where the system states live in.

*Assumption 3:* The system nonlinearities along with their Jacobian are well defined with respect to their arguments.

Starting from the first dynamical equation of (5), we have

$$\begin{aligned} x_1 &= y, \\ x_2 &= \dot{y} - g_1(y, u). \end{aligned} \quad (6)$$

From the second equation of (5), we extract the state  $x_3$  as

$$x_3 = \dot{x}_2 - g_2(y_1, x_2, u). \quad (7)$$

Substituting (6) in (7), we get

$$x_3 = \dot{x}_2 - g_2(y_1, \dot{y} - g_1(y, u), u). \quad (8)$$

By differentiating the second equation in (6) and substituting in (8), we get

$$\begin{aligned} x_3 &= \ddot{y} - \frac{\partial g_1}{\partial y}(y, u) \dot{y} - \frac{\partial g_1}{\partial u}(y, u) \dot{u} \\ &\quad - g_2(y_1, \dot{y} - g_1(y, u), u). \end{aligned} \quad (9)$$

The aforementioned system is observable for any input if the system nonlinearities  $g_1(s_1, u)$ ,  $g_2(s_1, s_2, u)$   $\frac{\partial g_1}{\partial y}(y, u)$ , and  $\frac{\partial g_1}{\partial u}(y, u)$  are non-singular and well-defined for all  $s_1, s_2 \in \mathbb{R}$  and  $t \geq 0$ .

### C. The second-order time-varying differentiator

Here, we show how to set up a model-free system that estimates the first and the second derivatives of any differentiable measured output. The differentiation observer is written in the controllable canonical form where the signal to be differentiated appears as a bounded input in the last dynamical equation of the time-derivative observer. The desired output derivatives are then given as nonlinear outputs of the differentiator states. The design is clarified in the following statement.

*Theorem 1:* Let  $y(t)$  be a  $\mathcal{C}^2$  continuous-time signal which is not necessarily bounded. Consider the time-varying system

$$\begin{aligned} \dot{\xi}_1(t) &= \xi_2(t), \\ \dot{\xi}_2(t) &= \xi_3(t), \\ \dot{\xi}_3(t) &= -\alpha^3 t^3 (\xi_1 - \arctan(y(t))) - 3\alpha^2 t^2 \xi_2(t) \\ &\quad - 3\alpha t \xi_3(t), \end{aligned} \quad (10)$$

where  $\alpha > 0$  is a constant. Then,

$$\begin{aligned} \lim_{t \rightarrow \infty} \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} &= \\ \lim_{t \rightarrow \infty} \begin{bmatrix} \tan(\xi_1(t)) \\ (1 + y^2(t)) \xi_2(t) \\ 2y(t)(1 + y^2(t)) \xi_2^2(t) + (1 + y^2(t)) \xi_3(t) \end{bmatrix}. \end{aligned} \quad (11)$$

*Proof:* To prove this result, it is sufficient to prove that  $\lim_{t \rightarrow \infty} (\xi_1(t) - \arctan(y(t))) = 0$ . By putting  $z(t) = \xi_1(t)$  then, the time-varying system (11) admits the following input-output representation:

$$\begin{aligned} z^{(3)}(t) + 3\alpha t \dot{z}(t) + 3\alpha^2 t^2 z(t) \\ + \alpha^3 t^3 (z(t) - \arctan(y(t))) = 0. \end{aligned} \quad (12)$$

By taking the following change of variable:  $z(t) = e^{-\frac{\alpha}{2} t^2} q(t)$  where  $q(t)$  is the new time-dependent variable then,  $q(t)$  verifies the following differential equation:

$$q^{(3)}(t) - 3\alpha \dot{q}(t) - \alpha^3 t^3 e^{\frac{\alpha}{2} t^2} \arctan(y(t)) = 0. \quad (13)$$

By solving (13) with respect to the variable  $q(t)$ , we get

$$\begin{aligned}
 q(t) &= C_3 + \int \left( C_1 e^{\sqrt{3\alpha}t} + C_2 e^{-\sqrt{3\alpha}t} \right) dt \\
 &+ \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} \int e^{\sqrt{3\alpha}t} \\
 &\times \int \left( t^3 \arctan(y(t)) e^{-\frac{1}{2}t(2\sqrt{3\alpha}-\alpha t)} dt \right) dt \\
 &- \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} \int e^{-\sqrt{3\alpha}t} \\
 &\times \int \left( t^3 \arctan(y(t)) e^{\frac{1}{2}t(2\sqrt{3\alpha}+\alpha t)} dt \right) dt.
 \end{aligned} \tag{14}$$

where  $C_1$ ,  $C_2$  and  $C_3$  are the integration constants. Let us note  $\hat{y}(t) = \arctan(y(t))$  then by returning back to the previous variable  $z(t)$ , we have

$$\begin{aligned}
 z(t) &= e^{-\frac{\alpha}{2}t^2} C_3 + e^{-\frac{\alpha}{2}t^2} \int \left( C_1 e^{\sqrt{3\alpha}t} + C_2 e^{-\sqrt{3\alpha}t} \right) dt \\
 &+ \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2}t^2} \int e^{\sqrt{3\alpha}t} \int \left( t^3 \hat{y}(t) e^{-\frac{1}{2}t(2\sqrt{3\alpha}-\alpha t)} dt \right) dt \\
 &- \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2}t^2} \int e^{-\sqrt{3\alpha}t} \int \left( t^3 \hat{y}(t) e^{\frac{1}{2}t(2\sqrt{3\alpha}+\alpha t)} dt \right) dt.
 \end{aligned} \tag{15}$$

For any  $\alpha > 0$ , we have

$$\begin{aligned}
 \lim_{t \rightarrow \infty} \left( e^{-\frac{\alpha}{2}t^2} C_3 + e^{-\frac{\alpha}{2}t^2} \int \left( C_1 e^{\sqrt{3\alpha}t} + C_2 e^{-\sqrt{3\alpha}t} \right) dt \right) \\
 = 0.
 \end{aligned} \tag{16}$$

This implies that

$$\begin{aligned}
 \lim_{t \rightarrow \infty} q(t) &= \lim_{t \rightarrow \infty} \left( \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2}t^2} \int e^{\sqrt{3\alpha}t} \times \right. \\
 &\times \int \left( t^3 \hat{y}(t) e^{-\frac{1}{2}t(2\sqrt{3\alpha}-\alpha t)} dt \right) dt \\
 &\left. - \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2}t^2} \int e^{-\sqrt{3\alpha}t} \int \left( t^3 \hat{y}(t) e^{\frac{1}{2}t(2\sqrt{3\alpha}+\alpha t)} dt \right) dt \right).
 \end{aligned} \tag{17}$$

Define

$$\begin{aligned}
 F_1(t) &= \int t^3 \hat{y}(t) e^{-\frac{1}{2}t(2\sqrt{3\alpha}-\alpha t)} dt, \\
 F_2(t) &= \int t^3 \hat{y}(t) e^{\frac{1}{2}t(2\sqrt{3\alpha}+\alpha t)} dt.
 \end{aligned} \tag{18}$$

Let

$$\begin{aligned}
 \psi_1(t) &= \int t^3 e^{-\frac{1}{2}t(2\sqrt{3\alpha}-\alpha t)} dt, \\
 &= \frac{(\alpha t^2 + \sqrt{3\alpha}t + 1) e^{-\frac{1}{2}t(2\sqrt{3\alpha}-\alpha t)}}{\alpha^2} \\
 \psi_2(t) &= \int t^3 e^{\frac{1}{2}t(2\sqrt{3\alpha}+\alpha t)} dt \\
 &= \frac{(\alpha t^2 - \sqrt{3\alpha}t + 1) e^{\frac{1}{2}t(2\sqrt{3\alpha}+\alpha t)}}{\alpha^2}.
 \end{aligned} \tag{19}$$

Integrating  $F_1(t)$  and  $F_2(t)$  by parts, we get

$$\begin{aligned}
 F_1(t) &= \psi_1 \hat{y}(t) - \int \dot{\hat{y}}(t) \psi_1 dt, \\
 F_2(t) &= \psi_2 \hat{y}(t) - \int \dot{\hat{y}}(t) \psi_2 dt.
 \end{aligned} \tag{20}$$

Then, using (17) and (20), we can write

$$\begin{aligned}
 \lim_{t \rightarrow \infty} z(t) &= \lim_{t \rightarrow \infty} \left[ \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2}t^2} \int e^{\sqrt{3\alpha}t} \psi_1 \hat{y}(t) dt \right. \\
 &- \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2}t^2} \int e^{-\sqrt{3\alpha}t} \psi_2 \hat{y}(t) dt \\
 &+ \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2}t^2} \int e^{\sqrt{3\alpha}t} \int \dot{\hat{y}}(t) \psi_1 dt dt \\
 &\left. - \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2}t^2} \int e^{-\sqrt{3\alpha}t} \int \dot{\hat{y}}(t) \psi_2 dt dt \right].
 \end{aligned} \tag{21}$$

Let

$$\begin{aligned}
 F_3(t) &= \int e^{\sqrt{3\alpha}t} \psi_1 \hat{y}(t) dt, \\
 F_4(t) &= \int e^{-\sqrt{3\alpha}t} \psi_2 \hat{y}(t) dt
 \end{aligned} \tag{22}$$

Then, by integration by parts, we get

$$\begin{aligned}
 F_3(t) &= \phi_1 \hat{y}(t) - \int \dot{\hat{y}}(t) \phi_1 dt, \\
 F_4(t) &= \phi_2 \hat{y}(t) - \int \dot{\hat{y}}(t) \phi_2 dt.
 \end{aligned} \tag{23}$$

where

$$\begin{aligned}
 \phi_1(t) &= \int e^{\sqrt{3\alpha}t} \psi_1 dt = \frac{e^{\frac{\alpha}{2}t^2} (\sqrt{\alpha t} + \sqrt{3})}{\alpha^{\frac{5}{2}}}, \\
 \phi_2(t) &= \int e^{-\sqrt{3\alpha}t} \psi_2 dt = \frac{e^{\frac{\alpha}{2}t^2} (\sqrt{\alpha t} - \sqrt{3})}{\alpha^{\frac{5}{2}}}.
 \end{aligned} \tag{24}$$

Remark that

$$\left( \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} \phi_1(t) - \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} \phi_2(t) \right) \hat{y}(t) = \hat{y}(t). \tag{25}$$

This gives

$$\begin{aligned}
 \lim_{t \rightarrow \infty} z(t) &= \lim_{t \rightarrow \infty} \left[ \arctan(y(t)) \right. \\
 &- \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2}t^2} \int e^{\sqrt{3\alpha}t} \int \dot{\hat{y}}(t) \phi_1(t) dt dt \\
 &+ \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2}t^2} \int e^{-\sqrt{3\alpha}t} \int \dot{\hat{y}}(t) \phi_2(t) dt dt \\
 &- \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2}t^2} \int e^{\sqrt{3\alpha}t} \int \dot{\hat{y}}(t) \psi_1 dt dt \\
 &\left. - \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2}t^2} \int e^{-\sqrt{3\alpha}t} \int \dot{\hat{y}}(t) \psi_2 dt dt \right].
 \end{aligned} \tag{26}$$

Using the result of Lemma 1, we can always find  $t = T > 0$  such that

$$\int e^{\frac{\alpha}{2}t^2} \leq e^{\frac{\alpha}{2}T^2}, \quad t \geq T. \tag{27}$$

Furthermore,

$$\begin{aligned} \int \phi_1(t) dt &= \frac{1}{\alpha^3} e^{\frac{\alpha}{2} t^2} + \frac{1}{\alpha^3} \int e^{\frac{\alpha}{2} t^2}, \\ \int \phi_2(t) dt &= \frac{1}{\alpha^3} e^{\frac{\alpha}{2} t^2} - \frac{\sqrt{3\alpha}}{\alpha^3} \int e^{\frac{\alpha}{2} t^2} \end{aligned} \quad (28)$$

Since  $\hat{y}(t)$  is globally bounded then,

$$\begin{aligned} &\lim_{t \rightarrow \infty} \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2} t^2} \int e^{\sqrt{3\alpha} t} \int \dot{y}(t) \phi_1(t) dt dt \\ &\leq \lim_{t \rightarrow \infty} \frac{\sqrt{3}}{6\sqrt{\alpha}} \sup_{t \geq 0} |\dot{y}(t)| e^{-\frac{\alpha}{2} t^2} \int e^{\sqrt{3\alpha} t} \left( e^{\frac{\alpha}{2} t^2} + \int e^{\frac{\alpha}{2} t^2} \right) \\ &= \lim_{t \rightarrow \infty} \frac{\sqrt{3}}{3\sqrt{\alpha}} \sup_{t \geq 0} |\dot{y}(t)| e^{-\frac{\alpha}{2} t^2} \int e^{\frac{\alpha}{2} t^2 + \sqrt{3\alpha} t} dt \\ &= \lim_{t \rightarrow \infty} \frac{\sqrt{3}}{3\sqrt{\alpha}} \sup_{t \geq 0} |\dot{y}(t)| e^{-\frac{\alpha}{2} t^2} \int e^{\frac{\alpha}{2} t^2} dt = 0. \end{aligned} \quad (29)$$

From (24), we have  $\phi_1(t) \geq \phi_2(t) \forall t$ . Taking into account (28), we have

$$\begin{aligned} &\lim_{t \rightarrow \infty} \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2} t^2} \int e^{-\sqrt{3\alpha} t} \int \dot{y}(t) \phi_2(t) dt dt \\ &\leq \lim_{t \rightarrow \infty} \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2} t^2} \int e^{-\sqrt{3\alpha} t} \int \dot{y}(t) \phi_1(t) dt dt \\ &\leq \lim_{t \rightarrow \infty} \frac{\sqrt{3}}{6\sqrt{\alpha}} \sup_{t \geq 0} |\dot{y}(t)| e^{-\frac{\alpha}{2} t^2} \int e^{-\sqrt{3\alpha} t} \left( e^{\frac{\alpha}{2} t^2} + \int e^{\frac{\alpha}{2} t^2} \right) \\ &\leq \lim_{t \rightarrow \infty} \frac{\sqrt{3}}{3\sqrt{\alpha}} \sup_{t \geq 0} |\dot{y}(t)| e^{-\frac{\alpha}{2} t^2} \int e^{\frac{\alpha}{2} t^2 - \sqrt{3\alpha} t} dt = 0. \end{aligned} \quad (30)$$

Consequently,

$$\begin{aligned} \lim_{t \rightarrow \infty} z(t) &= \lim_{t \rightarrow \infty} \left[ \arctan(y(t)) \right. \\ &\quad - \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2} t^2} \int e^{\sqrt{3\alpha} t} \int \dot{y}(t) \psi_1 dt dt \\ &\quad \left. - \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2} t^2} \int e^{-\sqrt{3\alpha} t} \int \dot{y}(t) \psi_2 dt dt \right]. \end{aligned} \quad (31)$$

Similarly, since  $\psi_1(t) > 0$

$$\begin{aligned} &-\frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2} t^2} \int e^{\sqrt{3\alpha} t} \int \dot{y}(t) \psi_1 dt dt \\ &-\frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{2} t^2} \int e^{-\sqrt{3\alpha} t} \int \dot{y}(t) \psi_2 dt dt \\ &\leq \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} \sup_{t \geq 0} |\dot{y}(t)| e^{-\frac{\alpha}{2} t^2} \int e^{\sqrt{3\alpha} t} \int \psi_1 dt dt \\ &+ \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} \sup_{t \geq 0} |\dot{y}(t)| e^{-\frac{\alpha}{2} t^2} \int e^{-\sqrt{3\alpha} t} \left| \int \psi_2 dt \right| dt \end{aligned} \quad (32)$$

From (19), we can find two positive constants  $C_\alpha$  and  $\bar{C}_\alpha$  such that

$$\begin{aligned} \int \psi_1(t) dt &= \frac{1}{\alpha^{\frac{5}{2}}} (\sqrt{\alpha} t + 2\sqrt{3}) e^{-\frac{t}{2}(2\sqrt{3\alpha} - \alpha t)} \\ &+ \frac{6\sqrt{\alpha}}{\alpha^{\frac{5}{2}}} \int e^{\frac{t}{2}(-2\sqrt{3\alpha} + \alpha t)} dt \\ &\leq C_\alpha e^{-\frac{t}{2}(2\sqrt{3\alpha} - \alpha t)} + \bar{C}_\alpha \int e^{\frac{t}{2}(-2\sqrt{3\alpha} + \alpha t)} dt. \end{aligned} \quad (33)$$

On the other hand, we have

$$\begin{aligned} \left| \int \psi_2(t) dt \right| &= \left| \frac{1}{\alpha^{\frac{5}{2}}} (\sqrt{\alpha} t - 2\sqrt{3}) e^{\frac{t}{2}(2\sqrt{3\alpha} + \alpha t)} \right. \\ &+ \left. \frac{6\sqrt{\alpha}}{\alpha^{\frac{5}{2}}} \int e^{\frac{t}{2}(2\sqrt{3\alpha} + \alpha t)} dt \right| \\ &\leq K_\alpha e^{\frac{t}{2}(2\sqrt{3\alpha} + \alpha t)} + \bar{K}_\alpha \int e^{\frac{t}{2}(2\sqrt{3\alpha} + \alpha t)} dt. \end{aligned} \quad (34)$$

Based on (34), we can conclude by the use of Lemma 1 that

$$\begin{aligned} &\lim_{t \rightarrow \infty} \left( \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} \sup_{t \geq 0} |\dot{y}(t)| e^{-\frac{\alpha}{2} t^2} \int e^{\sqrt{3\alpha} t} \int \psi_1 dt dt \right. \\ &+ \left. \frac{\sqrt{3}}{6} \alpha^{\frac{5}{2}} \sup_{t \geq 0} |\dot{y}(t)| e^{-\frac{\alpha}{2} t^2} \int e^{-\sqrt{3\alpha} t} \left| \int \psi_2 dt \right| dt \right) = 0. \end{aligned} \quad (35)$$

Finally, we can write that  $\lim_{t \rightarrow \infty} (\xi_1 - \hat{y}(t)) = 0$ . From the dynamics of (10), we conclude that  $\lim_{t \rightarrow \infty} (\xi_i - \hat{y}^{(i-1)}(t)) = 0$ ,  $2 \leq i \leq n$ . Consequently, (11) is verified. This ends the proof.

Remark that the differentiator states are given as multiple integrals of a time-varying combination of the differentiator states. For the null initial conditions, i.e.,  $\xi_i(0) = 0$ ,  $1 \leq i \leq 3$ , we have

$$\begin{aligned} \xi_1(t) &= \int_0^t \int_0^t \int_0^t \left( -\alpha^3 t^3 (\xi_1 - \arctan(y(t))) \right. \\ &\quad \left. - 3\alpha^2 t^2 \xi_2(t) - 3\alpha t \xi_3(t) \right) dt, \\ \xi_2(t) &= \int_0^t \int_0^t \left( -\alpha^3 t^3 (\xi_1 - \arctan(y(t))) \right. \\ &\quad \left. - 3\alpha^2 t^2 \xi_2(t) - 3\alpha t \xi_3(t) \right) dt, \\ \xi_3(t) &= \int_0^t \left( -\alpha^3 t^3 (\xi_1 - \arctan(y(t))) \right. \\ &\quad \left. - 3\alpha^2 t^2 \xi_2(t) - 3\alpha t \xi_3(t) \right) dt, \end{aligned} \quad (36)$$

which implies that the estimates states are robust with respect to measurement errors that may corrupt the bounded signal  $\arctan(y(t))$ . Actually, the differentiation observer gains, which are unbounded time-dependent, can be set bounded by adaptation. In other words, by setting the following

differentiation scheme

$$\begin{aligned}\dot{\xi}_1(t) &= \xi_2(t), \\ \dot{\xi}_2(t) &= \xi_3(t), \\ \dot{\xi}_3(t) &= -\theta^3(t)(\xi_1 - \arctan(y(t))) - 3\theta^2(t)\xi_2(t) \\ &\quad - 3\theta(t)\xi_3(t), \\ \dot{\theta}(t) &= \begin{cases} \alpha, & \text{if } |\xi_1 - \arctan(y(t))| \neq 0, \\ 0, & \text{if } |\xi_1 - \arctan(y(t))| = 0, \end{cases}\end{aligned}\quad (37)$$

the parameter  $\theta(t) = \alpha t$  when  $|\xi_1 - \arctan(y(t))| \neq 0$  and equal to some constant when  $\xi_1 = \arctan(y(t))$ . This adaptive scheme appends the sensitivity of the differentiator to measurements errors and allows the boundedness of the observer gain even the output to be differentiated may be unbounded.

*Remark 1:* The constant  $\alpha$  is a positive parameter that regulates the speed of convergence. For high-frequency outputs, it is recommended to chose  $\alpha$  sufficiently large in order to compensate the effects of fast changes in the signal direction and assure a fast convergence to the true derivatives.

### III. OBSERVER DESIGN

By exploiting the triangular structure of system (5) and the algebraic observability of all the unmeasured states, we show that the unmeasured states can be seen as nonlinear output of the developed time-varying differentiator. The rate of convergence of the estimates depends on the rate of convergence of the time-varying differentiator. This result is given in the following statement.

#### A. The nonlinear observer

Based on the previous developments, the analysis of the nonlinear algebraic observer is given in the following statement.

*Theorem 2:* Consider system (5) under assumptions 1-3. Define the nonlinear observer

$$\begin{aligned}\dot{\xi}_1 &= \xi_2, \dot{\xi}_2 = \xi_3, \\ \dot{\xi}_3 &= -\alpha^3 t^3(\xi_1 - \arctan(y)) - 3\alpha^2 t^2 \xi_2 - 3\alpha t \xi_3, \\ \hat{x}_1 &= \tan(\xi_1), \hat{x}_2 = \xi_2 - g_1(y, u), \\ \hat{x}_3 &= \xi_3 - \frac{\partial g_1}{\partial y}(y, u) \xi_2 - \frac{\partial g_1}{\partial u}(y, u) \dot{u} \\ &\quad - g_2(y_1, \xi_2 - g_1(y, u), u).\end{aligned}\quad (38)$$

Then, for any initial condition  $\hat{x}_0 \in \mathbb{R}^3$ , we have

$$\lim_{t \rightarrow \infty} (x_i - \hat{x}_i) = 0, \quad 1 \leq i \leq 3. \quad (39)$$

*Proof:* Using the result of Theorem 1 we conclude that the first and the second derivatives of  $\arctan(y(t))$  are reproduced asymptotically by  $\xi_2$  and  $\xi_3$ , respectively the static expression (6) and (9) derived from the system dynamics can be reproduced by replacing the true derivatives by their estimates. This ends the proof.

To illustrate the observer design, let us consider the following nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1 u, \quad \dot{x}_2 = x_3 + \frac{u}{1 + x_1^2 + x_2^2}, \\ \dot{x}_3 &= -x_3 + x_1 x_2 + u, \quad y = x_1.\end{aligned}\quad (40)$$

The system is algebraically observable since  $y = x_1$ ,  $x_2 = \dot{y} - y u$ ,  $x_3 = \dot{y} - \dot{y} u - y \dot{u} - u / (1 + y^2 + (\dot{y} - y u)^2)$ . Based on these algebraic expressions the nonlinear observer is readily constructed as

$$\begin{aligned}\dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= \xi_3, \\ \dot{\xi}_3 &= -\alpha^3 t^3(\xi_1 - \arctan(y)) - 3\alpha^2 t^2 \xi_2 - 3\alpha t \xi_3, \\ \hat{x}_1 &= \tan(\xi_1), \\ \hat{x}_2 &= \xi_2 - y u, \\ \hat{x}_3 &= \xi_3 - \xi_2 u - y \dot{u} - u / (1 + y^2 + (\xi_2 - y u)^2).\end{aligned}\quad (41)$$

#### B. Application to 2-link robotic manipulator

In fact, the nonlinear observer design proposed herein does not depend on the triangular structure of the system. To clarify this point let us consider the 2-link robotic manipulator model represented in Fig. 1 [23]. The dynamical equation of the robot are given by

$$\begin{aligned}\begin{bmatrix} D_{11}(\phi) & D_{12}(\phi) \\ D_{12}(\phi) & D_{22}(\phi) \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} &= \\ \begin{bmatrix} F_{12}(\phi) \dot{\phi}^2 + 2F_{12}(\phi) \dot{\theta} \dot{\phi} \\ -F_{12}(\phi) \dot{\theta}^2 \end{bmatrix} &+ \begin{bmatrix} q_1(\theta, \phi) g \\ q_2(\theta, \phi) g \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},\end{aligned}\quad (42)$$

where

$$\begin{aligned}D_{11}(\phi) &= (m_1 + m_2)r_1^2 + m_2 r_2^2 + 2m_2 r_1 r_2 \cos(\phi) + J_1, \\ D_{12}(\phi) &= m_2 r_2^2 + m_2 r_1 r_2 \cos(\phi), \\ D_{22}(\phi) &= m_2 r_2^2 + J_2, \\ F_{12}(\phi) &= m_2 r_1 r_2 \sin(\phi), \\ q_1(\theta, \phi) &= -\left( (m_1 + m_2)r_1 \cos(\phi) + m_2 r_2 \cos(\phi + \theta) \right), \\ q_2(\theta, \phi) &= -m_2 r_2 \cos(\theta + \phi).\end{aligned}\quad (43)$$

The values of the system parameters are:  $r_1 = 1$  m,  $r_2 = 0.8$  m,  $J_1 = 5$  Kg,  $J_2 = 5$  Kg,  $m_1 = 0.5$  Kg,  $m_2 = 6.26$  Kg. If we assume that  $\theta$  and  $\phi$  are measured then by putting  $x_1 = \theta$ ,  $x_2 = \dot{\theta}$ ,  $x_3 = \phi$ ,  $x_4 = \dot{\phi}$  then the dynamics of the robot is given by the following state-space representation:

$$\begin{aligned}\dot{x}_1 &= x_2, \dot{x}_2 = f_1(x, u), \dot{x}_3 = x_4, \dot{x}_4 = f_2(x, u), \\ y_1 &= x_1, y_2 = x_3,\end{aligned}\quad (44)$$

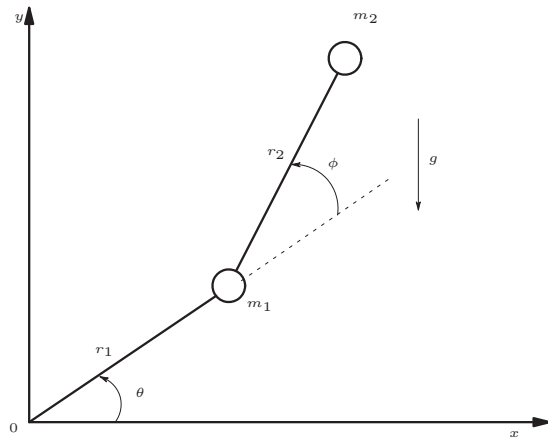


Fig. 1. The 2-link robot

where

$$\begin{bmatrix} f_1(x, u) \\ f_2(x, u) \end{bmatrix} = D^{-1}(x_3) \begin{bmatrix} F_{12}(x_3) x_4^2 + 2F_{12}(x_3) x_2 x_4 \\ -F_{12}(x_3) x_2^2 \end{bmatrix} \\ + D^{-1}(x_3) \begin{bmatrix} q_1(x_1, x_3) g \\ q_2(x_1, x_3) g \end{bmatrix} + D^{-1}(x_3) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \\ D(x_3) = \begin{bmatrix} D_{11}(x_3) & D_{12}(x_3) \\ D_{12}(x_3) & D_{22}(x_3) \end{bmatrix}. \quad (45)$$

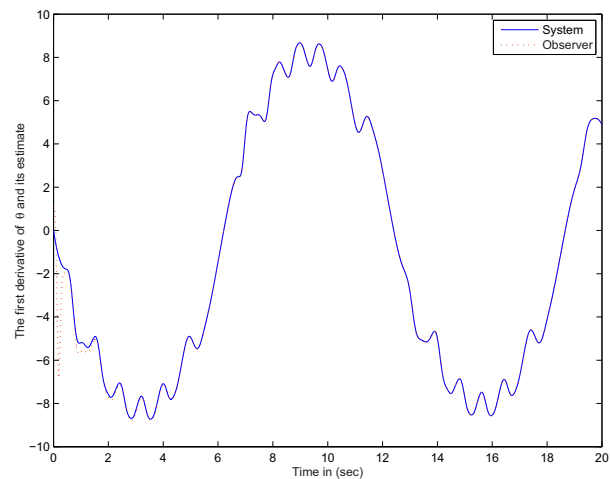
System (44) has not triangular structure, but the system is algebraically observable in the sense that the unmeasured state variables are given as the first derivatives of the measured variables, i.e.,  $x_2 = \dot{y}_1$ ,  $x_4 = \dot{y}_2$ . In Fig. 2, we have represented the estimated state the true state  $x_2$  and its estimate  $\hat{x}_2$  given by the algebraic observer when  $u_1 = -x_1$ ,  $u_2 = -x_3$ .

#### IV. CONCLUSION

In the paper, we showed that the uniform complete observability condition implies the existence of globally convergent observer for three-states nonlinear systems. The proposed technique enjoys the property of being independent on the form of the system nonlinearities whenever the algebraic observability condition is verified.

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Fig. 2. The true state  $\dot{\theta}$  and its estimate

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