Ann Oper Res (2003) 158: 133–141
DOI 10.1007/s10479-007-0241-x

Compromise solutions for bankruptcy situations with references

M. Pulido · P. Borm · R. Hendrickx · N. Llorca · J. Sánchez-Soriano

Published online: 13 September 2003

Abstract This paper deals with bankruptcy situations in which in addition to the claims, an exogenously given reference point for the allocation of the estate is present. We introduce and analyse two types of compromise solutions and show that they coincide with the τ value of two corresponding TU games. We apply our solutions to a real-life case of allocating university money to degree courses.

1 Introduction

Bankruptcy problems were first introduced by O'Neill (1982) and have been subsequently analysed in a variety of contexts. In a bankruptcy situation, one has to divide a given amount of money (the estate) among a set of agents, each of whom has a rightful claim on the estate. The total amount claimed typically exceeds the estate available, so not all the claims of the agents can be fully satisfied.

In some situations, however, the claims of the players are not the only quantities that are relevant for determining how to divide the estate. Pulido et al. (2002) analyse the problem

Research supported by Generalitat Valenciana, through ACOMP06/040, ACOMP07/136 and GV05/189, and by the Government of Spain and FEDER fund under project MTM2005-09184-C02-02. M. Pulido's research is partially supported by Fundación Séneca of Comunidad Autónoma de la Región de Murcia, through 02911/PI/05. Ruud Hendrickx author acknowledges financial support from the Netherlands Organisation for Scientific Research (NWO).

M. Pulido (✉)
Department of Statistics and Operations Research, Faculty of Mathematics, University of Murcia, Campus de Espinardo, 30100 Murcia, Spain
e-mail: mpulido@um.es

P. Borm · R. Hendrickx
Center and Department of Econometrics and Operations Research, Tilburg University, Tilburg, The Netherlands

N. Llorca · J. Sánchez-Soriano
Operations Research Center and Department of Statistics, Mathematics and Computer Science, Miguel Hernández University, Elche, Spain

of dividing a sum of money to the various degree courses that are offered at Miguel Hernández University in Elche, Spain. Each course has a claim, which reflects in some way the monetary *needs* of this course. These needs are determined within a fixed set of rules and are verifiable to everyone involved. In addition to these claims, the Valencian government (*Generalitat Valenciana*) provides a set of rules of its own to indicate what each course should get, without taking into account how much money is available. This allocation can be considered as an exogenous *reference point* for determining a fair division of the estate.

Clearly, both the claims and the references form a relevant basis for the allocation decision. The natural question is how the two sets of data in such a *bankruptcy situation with references* should be combined in order to reach a fair outcome. In Pulido et al. (2002), the special case is considered, in which the estate suffices to implement the reference point. In this case, the references can be interpreted as *rights*. They describe a two-stage procedure which first gives each claimant his reference amount and then shares the remainder using a bankruptcy rule.

In this paper, we consider situations in which the estate is not necessarily big enough to pay all reference amounts. Hence, we do not regard these references as rights. As a result, our analysis extends the analysis in Pulido et al. (2002), but we provide different answers on the class of situations in which both models are applicable.

We consider two ways in which the claim and reference vectors are combined and for either approach, we define a compromise solution. The underlying idea is the following: for each player we combine the claim and reference vectors in such a way that the resulting outcome to him is maximal given a benchmark bankruptcy rule. Doing this for every player, we obtain an upper vector, which can be seen as a utopia point. On the other hand, we find for each player that combination which gives him a minimal outcome given the same benchmark, which results in a lower vector. A compromise solution is then defined as the convex combination of the upper and lower vector that is efficient with respect to the estate. For both approaches we also define a corresponding *bankruptcy game with references*. These games are exact, but not necessarily convex. The compromise solutions turn out to coincide with the τ values (Tijs 1981) of these games.

To illustrate our compromise rules, we apply them to the university case that was also analysed in Pulido et al. (2002).

The paper is organised as follows. First, in Sect. 2 we introduce bankruptcy situations with references. In Sect. 3 we introduce a new monotonicity property for bankruptcy rules and present our compromise solutions. In Sect. 4 we define and analyse the two corresponding games and show that the compromise solutions coincide with their τ values. For the university case these solutions are presented in Sect. 5.

2 Bankruptcy with references

A *bankruptcy situation* (O'Neill 1982) is a triple $(N, E, \mathbf{d})$, where N is the finite set of agents, $E \geq 0$ is the estate to be divided and $\mathbf{d} \in \mathbb{R}_+^N$ is the vector of claims such that $\sum_{i\in N} d_i \geq E$. Every bankruptcy situation $(N, E, \mathbf{d})$ gives rise to a *bankruptcy game* (N, v), where the value of a coalition $S \subset N$ is given by

$$v(S) = \max\left\{E - \sum_{i\in N\setminus S} d_i, 0\right\}.$$

So $v(S)$ is the part of the estate which is left for the players in S after the claims of the players in $N\setminus S$ have been satisfied. Hence, in computing the value of a coalition, a pessimistic point of view is taken.

A *bankruptcy rule* is a function f that assigns to every bankruptcy situation $(N, E, \mathbf{d})$ a vector $f(N, E, \mathbf{d}) \in \mathbb{R}^N$ such that

1. $0 \leq f_i(N, E, \mathbf{d}) \leq d_i$ for all $i \in N$,
2. $\sum_{i \in N} f_i(N, E, \mathbf{d}) = E$.

A *bankruptcy situation with references* (Pulido et al. 2002) is a 4-tuple $(N, E, \mathbf{r}, \mathbf{c})$, where N is the finite set of agents, $E \geq 0$ is the estate under contest, $\mathbf{r} \in \mathbb{R}^N_+$ is the vector of references and the vector of claims, $\mathbf{c} \in \mathbb{R}^N_+$, is such that $\sum_{i \in N} c_i \geq E$. The claim vector $\mathbf{c}$ has the same interpretation as the vector $\mathbf{d}$ in standard bankruptcy situations, while $\mathbf{r}$ represents some exogenously given reference point for the division of the estate. We assume that $r_i \leq c_i$ for every player $i \in N$.

For a vector $\mathbf{x} \in \mathbb{R}^N$ and a coalition $S \subset N$, we denote by $\mathbf{x}_S$ the restricted vector $(x_i)_{i \in S}$ and $x(S) = \sum_{i \in S} x_i$. Furthermore, we denote $R = r(N)$ and $C = c(N)$.

Pulido et al. (2002) distinguish between two types of bankruptcy situations with references: *CERO-bankruptcy situations* ($C \geq E \geq R \geq 0$) and *CREO-bankruptcy situations* ($C \geq R > E \geq 0$). They only analyse the CERO case, in which the estate is sufficient to give each player his reference amount. So basically, such situation can be solved by first allocating this reference point and then dividing the surplus $E - R$. Using this idea, Pulido et al. (2002) define corresponding *CERO bankruptcy games*.

In this paper, we consider both cases simultaneously. The analysis differs from the CERO case, because in our context the references cannot necessarily all be satisfied and can therefore not be considered as rights. Hence, we take a different approach to solving such situations and defining appropriate corresponding games. Note that as a result, the analysis differs from Pulido et al. (2002) even on the class of CERO situations.

3 Compromise solutions

In this section, we define two (families of) compromise solutions for bankruptcy situations with references. In order to come to a solution we have to make some assumptions on the way in which the claims and reference point are used to divide the money. Obviously, the claim and reference vectors should both be taken into account, but this can be done in a number of ways. We assume that first a new combined demand vector is constructed, reflecting both $\mathbf{r}$ and $\mathbf{c}$. After that, a given bankruptcy rule f is applied to this new vector. This rule f reflects some benchmark rule for evaluating claims and can be interpreted, e.g., as the method that was used on a previous occasion to solve a similar bankruptcy problem.

We assume that f satisfies ***complementary monotonicity (CM)***: for all $S \subset N$ and $\mathbf{d}, \mathbf{d}' \in \mathbb{R}^N_+$ such that $d_j = d'_j$, for all $j \in N \backslash S$ and $d_i \leq d'_i$ for all $i \in S$, we have that $f_j(N, E, \mathbf{d}) \geq f_j(N, E, \mathbf{d}')$ for all $j \in N \backslash S$. Using an induction argument, it is easily established that this is equivalent with the same requirement only for all one-person coalitions S.

Most well-known bankruptcy rules satisfy (CM), as is shown in Pulido (2001). One notable exception is the adjusted proportional rule. In this paper, we use two bankruptcy rules that are (CM): the *proportional rule*, which distributes the estate proportional to the claims, and the *constrained equal award* (*CEA*) *rule*, which allocates $\min\{\alpha, d_i\}$ to claimant i, where α is uniquely determined by efficiency.

We solve a bankruptcy situation with references by means of a compromise solution. Given the benchmark f, we determine for each player which combination of references and claims leads to the highest outcome for him and which to his lowest outcome. This leads to an upper and lower bound for the allocation of the estate. The compromise solution is then simply defined as the unique efficient convex combination of these two vectors.

Geometrically, combining claims and references boils down to picking a point in the hypercube $\Pi_{i\in N}[r_i, c_i]$. We consider two approaches. In our first approach, the **extreme approach**, we consider the extreme points of this hypercube, i.e., points in which some players demand their reference amount and the others their claim. The lower vector $\mathbf{l}^f$ and the upper vector $\mathbf{L}^f$ are defined by

$$l_i^f(N, E, \mathbf{r}, \mathbf{c}) = \begin{cases} f_i(N, E, (r_i, \mathbf{c}_{N\setminus\{i\}})) & \text{if } r_i + c(N\setminus\{i\}) \geq E, \\ E - c(N\setminus\{i\}) & \text{if } r_i + c(N\setminus\{i\}) < E, \end{cases}$$

$$L_i^f(N, E, \mathbf{r}, \mathbf{c}) = \begin{cases} f_i(N, E, (c_i, \mathbf{r}_{N\setminus\{i\}})) & \text{if } c_i + r(N\setminus\{i\}) \geq E, \\ c_i & \text{if } c_i + r(N\setminus\{i\}) < E, \end{cases}$$

for all $i \in N$. It follows from CM that $(r_i, \mathbf{c}_{N\setminus\{i\}})$ is the worst extreme point for player i and that $(c_i, \mathbf{r}_{N\setminus\{i\}})$ is the best. If in a point the estate suffices to satisfy all demands, then player i gets what is left by the other players, with a maximum of his own claim c_i.

In the following lemma, which we will prove in Sect. 4, we show that $\mathbf{l}^f$ and $\mathbf{L}^f$ can indeed be considered as lower and upper bounds, respectively, for the division of the estate.

Lemma 1 *Let $(N, E, \mathbf{r}, \mathbf{c})$ be a bankruptcy situation with references. Then for the corresponding lower and upper vectors we have $\mathbf{l}^f \leq \mathbf{L}^f$ and $\sum_{i\in N} l_i^f \leq E \leq \sum_{i\in N} L_i^f$.*

The *extreme compromise solution* γ^f is defined by

$$\gamma^f = \alpha \mathbf{l}^f + (1-\alpha)\mathbf{L}^f,$$

where $\alpha \in [0, 1]$ is such that $\sum_{i\in N}(\alpha {l_i}^f + (1-\alpha){L_i}^f) = E$. As a result of the previous lemma, such α exists.

In our second approach, the **diagonal approach**, we consider the main diagonal of the hypercube. The lower and upper vectors $\bar{\mathbf{l}}^f$ and $\bar{\mathbf{L}}^f$ are defined by

$$\bar{l}_i{}^f(N, E, \mathbf{r}, \mathbf{c}) = \inf_{\lambda\in[0,1]} h_i^{f,\lambda}(N, E, \mathbf{r}, \mathbf{c}),$$

$$\bar{L}_i{}^f(N, E, \mathbf{r}, \mathbf{c}) = \sup_{\lambda\in[0,1]} h_i^{f,\lambda}(N, E, \mathbf{r}, \mathbf{c}),$$

where for all $i \in N$,

$$h_i^{f,\lambda}(N, E, \mathbf{r}, \mathbf{c}) = \begin{cases} f_i(N, E, \lambda\mathbf{r} + (1-\lambda)\mathbf{c}) & \text{if } \lambda R + (1-\lambda)C \geq E, \\ \lambda r_i + (1-\lambda)c_i + f_i(N, \overline{E}^\lambda, \overline{\mathbf{d}}^\lambda) & \text{if } \lambda R + (1-\lambda)C < E \end{cases}$$

with $\overline{E}^\lambda = E - (\lambda R + (1-\lambda)C)$ and $\overline{\mathbf{d}}^\lambda = \mathbf{c} - (\lambda\mathbf{r} + (1-\lambda)\mathbf{c}) = \lambda(\mathbf{c} - \mathbf{r})$. In the second part of the definition the agents receive what is prescribed by the weighted vector and the remainder $\bar{E}^\lambda$ is distributed according to the rule f, using the residual claims $\bar{\mathbf{d}}^\lambda$.

Also in the diagonal case, the vectors $\bar{\mathbf{l}}$ and $\bar{\mathbf{L}}$ can be considered as lower and upper bounds, as was shown for the extreme approach in Lemma 1.

The *diagonal compromise solution* $\bar{\gamma}^f$ is defined by

$$\bar{\gamma}^f = \alpha\bar{\mathbf{l}}^f + (1-\alpha)\bar{\mathbf{L}}^f,$$

where $\alpha \in [0, 1]$ is such that $\sum_{i\in N}(\alpha\bar{l}_i{}^f + (1-\alpha)\bar{L}_i{}^f) = E$.

4 Bankruptcy games with references

In this section, we define for either approach a corresponding cooperative game, depending on a bankruptcy rule f (which we assume to satisfy CM). The usual procedure in cooperative game theory is first to define the value of each coalition and then as a second step apply a solution concept to find an allocation. Note that in our specific setting, the rule f in the definition of the game should not be interpreted as a proposal for dividing the value of the grand coalition (i.e., the estate). Rather, as in the definitions of the upper and lower vectors in the previous section, it represents some benchmark rule for evaluating claims, which each coalition uses to compute a fair estimate of its worth. So contrary to standard bankruptcy games, the uncertainty for a coalition here is not the way in which the claims are processed (in a bankruptcy game, each coalition assumes that it is served last), but the way in which the claims and references are combined.

We define the *extreme game* (N, v^f) by

$$v^f(S) = \begin{cases} \sum_{i \in S} f_i(N, E, (\mathbf{r}_S, \mathbf{c}_{N\setminus S})) & \text{if } r(S) + c(N\setminus S) \geq E, \\ E - c(N \setminus S) & \text{if } r(S) + c(N\setminus S) < E \end{cases}$$

for all $S \subset N$. Note that if $r(S) + c(N\setminus S) < E$, the ensuing problem is not a bankruptcy situation and the players in S can obtain what is left by the players in $N\setminus S$.

As a result of CM, it immediately follows that $(\mathbf{r}_S, \mathbf{c}_{N\setminus S})$ is the worst point for S in the hypercube, thus $v^f(S)$ actually represents the most pessimistic situation for coalition S under the extreme approach. Although it is intuitively clear that this should be the worst point for S, we need CM to ensure that it is actually so.[1]

The *diagonal game* is defined by $\bar{v}^f(S) = \inf_{\lambda \in [0,1]} \bar{v}^f_\lambda(S)$, where

$$\bar{v}^f_\lambda(S) = \begin{cases} \sum_{i \in S} f_i(N, E, \lambda \mathbf{r} + (1-\lambda)\mathbf{c}) & \text{if } \lambda R + (1-\lambda)C \geq E, \\ \lambda r(S) + (1-\lambda)c(S) + \sum_{i \in S} f_i(N, \overline{E}^\lambda, \overline{\mathbf{d}}^\lambda) & \text{if } \lambda R + (1-\lambda)C < E, \end{cases}$$

with $\overline{E}^\lambda = E - (\lambda R + (1-\lambda)C)$ and $\overline{\mathbf{d}}^\lambda = \mathbf{c} - (\lambda \mathbf{r} + (1-\lambda)\mathbf{c}) = \lambda(\mathbf{c} - \mathbf{r})$.

Note that if $(N, E, \mathbf{r}, \mathbf{c})$ is a CREO-bankruptcy situation, then in the definitions of both v^f and $\bar{v}^f$ only the first case arises.

As a result of CM, it is easy to check that the game v^f is more pessimistic than $\bar{v}^f$, i.e., $v^f(S) \leq \bar{v}^f(S)$ for all $S \subset N$.

A game (N, v) is *convex* (cf. Shapley 1971) if

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$$

for all $S, T \subset N$. The *core* of a game (N, v) is defined by

$$C(v) = \{\mathbf{x} \in \mathbb{R}^N \mid x(N) = v(N), \forall_{S \subset N} : x(S) \geq v(S)\}.$$

A game (N, v) is *exact* (Driessen and Tijs 1985) if for all $S \subset N$ there exists an $\mathbf{x} \in C(v)$ such that $x(S) = v(S)$. Exactness is a weaker property than convexity. The extreme and diagonal games corresponding to a bankruptcy situation with references turn out to be exact, as is stated in the following proposition.

[1] Pulido (2001) shows that for the adjusted proportional rule, which is not CM, this is not the case.

Proposition 1 *Let $(N, E, \mathbf{r}, \mathbf{c})$ be a bankruptcy situation with references. Then the two corresponding games v^f and $\bar{v}^f$ are exact.*

Proof First, we prove exactness of v^f. Let $S \subset N$ and distinguish between two cases:

1. If $r(S) + c(N/S) \geq E$, then $v^f(S) = \sum_{i \in S} f_i(N, E, (\mathbf{r}_S, \mathbf{c}_{N\backslash S}))$. Consider $\mathbf{x} = f(N, E, (\mathbf{r}_S, \mathbf{c}_{N\backslash S}))$, so $x(S) = v^f(S)$. It is easy to check that, because f is CM, $\mathbf{x} \in C(v^f)$.
2. If $r(S) + c(N/S) < E$, then $v^f(S) = E - c(N\backslash S)$. Let $\mathbf{x} \in \mathbb{R}^N$ be defined by

$$x_i = \begin{cases} c_i & \text{if } i \in N\backslash S, \\ r_i + f_i(S, E - c(N\backslash S) - r(S), \mathbf{c}_S - \mathbf{r}_S) & \text{if } i \in S, \end{cases}$$

for all $i \in N$.
Obviously, $x(N) = E = v^f(N)$ and $x(S) = E - c(N\backslash S) = v^f(S)$. Let $T \subset N$ and distinguish between two cases:

(a) If $r(T) + c(N/T) \geq E$, then $v^f(T) = \sum_{i \in T} f_i(N, E, (\mathbf{r}_T, \mathbf{c}_{N\backslash T})) \leq r(T) \leq x(T)$.
(b) If $r(T) + c(N/T) < E$, then $v^f(T) = E - c(N\backslash T)$. Define $T_1 = T \cap (N\backslash S)$ and $T_2 = T \cap S$. Then,

$$\begin{aligned} x(T) &= c(T_1) + r(T_2) + \sum_{i \in T_2} f_i(S, E - c(N\backslash S) - r(S); \mathbf{c}_S - \mathbf{r}_S) \\ &\geq c(T_1) + r(T_2) + E - c(N\backslash S) - r(S) - \sum_{i \in S\backslash T_2} (c_i - r_i) \\ &= E + c(T_1) - c(N\backslash S) - c(S\backslash T_2) + r(T_2) + r(S\backslash T_2) - r(S) \\ &= E + c(T_1) - c(N\backslash T_2) = E - c(N\backslash T) = v^f(T). \end{aligned}$$

Hence, $\mathbf{x} \in C(v^f)$.

To show that $\bar{v}^f$ is exact, let $S \subset N$ and let $\lambda^* \in [0, 1]$ be such that $\bar{v}^f(S) = \inf_{\lambda \in [0,1]} \bar{v}^f_\lambda(S) = \bar{v}^f_{\lambda^*}(S)$. We distinguish between two cases:

1. If $\lambda^* R + (1 - \lambda^*)C \geq E$, then $\mathbf{x} = f(N, E, \lambda^* \mathbf{r} + (1 - \lambda^*)\mathbf{c}) \in C(\bar{v}^f)$ and $x(S) = \bar{v}^f(S)$.
2. If $\lambda^* R + (1 - \lambda^*)C < E$, then $\mathbf{x} = \lambda^* \mathbf{r} + (1 - \lambda^*)\mathbf{c} + f(N, E - \lambda^* R + (1 - \lambda^*)C, \lambda^*(\mathbf{c} - \mathbf{r})) \in C(\bar{v}^f)$ and $x(S) = \bar{v}^f(S)$.

Note that complementary monotonicity of f is not necessary to establish exactness of $\bar{v}^f$. □

The next example shows that the games v^f and $\overline{v}^f$ need not be convex.

Example 1 Consider the bankruptcy situation with references $(N, E, \mathbf{r}, \mathbf{c})$, where $N = \{1, 2, 3, 4\}$, $E = 12$, $\mathbf{r} = (1, 2, 4, 7)$ and $\mathbf{c} = (3, 3, 5, 10)$. Taking the CEA rule, we obtain

S	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v^{CEA}(S)$	3	5	$5\frac{1}{2}$	7
$\overline{v}^{CEA}(S)$	3	5	6	7

Then,

$$v^{CEA}(\{1,3\}) + v^{CEA}(\{2,3\}) > v^{CEA}(\{3\}) + v^{CEA}(\{1,2,3\}),$$
$$\overline{v}^{CEA}(\{1,3\}) + \overline{v}^{CEA}(\{2,3\}) > \overline{v}^{CEA}(\{3\}) + \overline{v}^{CEA}(\{1,2,3\}).$$

Therefore, v^{CEA} and $\overline{v}^{CEA}$ are not convex.

Using exactness of v^f, we can now prove Lemma 1.

Proof of Lemma 1 From complementary monotonicity of f, $\mathbf{l}^f \leq \mathbf{L}^f$ readily follows. For the second statement, first observe that $l_i^f = v^f(\{i\})$ for all $i \in N$. Then exactness of v^f implies

$$\sum_{i \in N} l_i^f = \sum_{i \in N} v^f(\{i\}) \leq v^f(N) = E$$

If $i \in N$ is such that $c_i + r(N \backslash \{i\}) < E$, then $L_i^f = c_i \geq f_i(N, E, \mathbf{c})$. On the other hand, if $c_i + r(N \backslash \{i\}) \geq E$, then $L_i^f = f_i(N, E, (c_i, \mathbf{r}_{N \backslash \{i\}})) \geq f_i(N, E, \mathbf{c})$. Combining these two cases, we have $\sum_{i \in N} L_i^f \geq \sum_{i \in N} f_i(N, E, \mathbf{c}) = E$. □

The τ *value* (Tijs 1981) of a quasi-balanced game (N, v) is defined by

$$\tau(v) = \alpha \mathbf{m}(v) + (1 - \alpha) \mathbf{M}(v),$$

where $M_i(v) = v(N) - v(N \setminus \{i\})$, $m_i(v) = \max_{S: i \in S} \{v(S) - \sum_{j \in S \backslash \{i\}} M_j(v)\}$ for all $i \in N$ and $\alpha \in [0, 1]$ is such that $\sum_{i \in N} \tau_i(v) = v(N)$. The class of quasi-balanced games contains the class of exact games, so as a result of Proposition 1, the τ value is defined for both bankruptcy games with references.

The extreme compromise solution coincides with the τ value of the game v^f, as is stated in the following theorem.

Theorem 1 *Let* $(N, E, \mathbf{r}, \mathbf{c})$ *be a bankruptcy situation with references and let* f *be a complementary monotonic bankruptcy rule. Then* $\gamma^f = \tau(v^f)$.

Proof Driessen and Tijs (1985) proved that for each exact game (N, v), $m_i(v) = v(\{i\})$ for all $i \in N$. Hence, $m_i(v^f) = v^f(\{i\}) = l_i^f$. For the upper vector, we have

$$\begin{aligned} L_i^f &= f_i(N, E, (c_i, \mathbf{r}_{N \backslash \{i\}})) \\ &= E - \sum_{j \in N \backslash \{i\}} f_j(N, E, (c_i, \mathbf{r}_{N \backslash \{i\}})) \\ &= v^f(N) - v^f(N \backslash \{i\}) \end{aligned}$$

if $c_i + r(N \backslash \{i\}) \geq E$ and

$$\begin{aligned} L_i^f &= c_i \\ &= E - (E - c_i) \\ &= v^f(N) - v^f(N \backslash \{i\}) \end{aligned}$$

otherwise. Hence, $\mathbf{L}^f = \mathbf{M}(v^f)$. From this, we conclude that $\gamma^f = \tau(v^f)$. □

Similarly, one can prove the analogous result for the diagonal compromise solution.

Theorem 2 *Let* $(N, E, \mathbf{r}, \mathbf{c})$ *be a bankruptcy situation with references and let* f *be a complementary monotonic bankruptcy rule. Then* $\bar{\gamma}^f = \tau(\bar{v}^f)$.

Table 1 Compromise solutions

Degree	**r**	**c**	γ^{CEA}	$\overline{\gamma}^{CEA}$	γ^{PROP}	$\overline{\gamma}^{PROP}$
1	3500.29	15,720.66	4635.16	9353.95	3106.79	3652.28
2	3164.85	25,532.20	5242.03	9735.25	4675.68	4778.88
3	17,720.24	32,960.44	19,135.55	24,721.18	8045.22	12,406.70
4	2483.86	13,664.61	3522.18	7839.54	2618.07	2919.05
5	2786.77	8173.76	3287.04	5367.19	1757.09	2341.26
6	3904.17	3904.17	3904.17	3904.17	1216.96	2295.41
7	5452.38	14,869.04	6326.87	9963.04	3253.47	4434.20
8	21,439.30	289,753.13	46,356.69	39,197.91	51,128.33	47,610.37
9	24,384.26	250,962.13	45,425.78	38,863.55	45,156.26	43,896.77
10	21,203.71	126,857.13	31,015.38	34,468.04	24,106.11	26,250.47
11	19,350.35	248,338.50	40,615.69	37,853.46	43,942.80	41,251.60
12	38,284.47	227,091.64	50,184.29	42,268.85	43,325.97	47,141.50
13	3769.55	63,069.72	9276.55	19,901.55	10,957.08	9952.82
14	3937.83	15,915.98	5050.20	9675.47	3203.41	3877.92
15	2692.53	10,059.72	3376.70	6221.48	2054.27	2544.20
16	37,595.35	530,070.44	61,600.06	52,478.51	76,116.29	86,354.18
17	27,882.87	121,229.15	36,551.62	36,198.29	24,172.70	28,571.76
18	31,758.44	233,163.45	50,462.24	40,312.52	43,337.49	44,948.17
19	33,548.98	248,008.45	52,126.76	41,379.22	46,078.44	47,704.03
20	3251.24	169,534.83	18,693.42	25,793.58	28,440.75	23,605.61
21	24,173.91	240,404.84	44,254.54	38,559.43	43,373.41	42,423.19
22	27,861.67	250,845.44	48,569.41	39,732.68	45,673.83	45,472.37
23	5041.77	70,752.96	11,144.14	21,177.15	12,414.27	11,537.22
24	4099.38	140,679.05	16,783.07	25,314.25	23,809.86	20,228.96
25	14,065.13	227,684.44	33,903.22	33,582.56	39,707.35	36,139.13
26	12,627.99	234,125.14	33,197.67	32,717.70	40,550.14	36,321.96
27	8834.88	264,726.58	32,598.67	30,658.59	45,017.09	38,579.11
Total	404,816.17	4078,097.61	717,239.11	717,239.11	717,239.11	717,239.11

5 Case

In this section we consider the case of dividing university money as described in Pulido et al. (2002). The problem is as follows: in the academic year 2000–2001, Miguel Hernández University in Elche, Spain, had 717,239.11 euros to divide among the 27 different degree courses to buy equipment for teaching laboratories. The board of the university decided to create a committee to study how to allocate the money. This committee asked each teaching unit what their real needs were and added these demands for each degree course. Therefore, these demands represent the claims of each administrative entity (degree course representative) and they were based on reports on the requirements for the next academic period.

Since these demands could be biased (due to way in which they were obtained) and could include a high level of subjectivity, the committee looked for a more objective measure for reference. In this case, these reference amounts were obtained by applying objective criteria and laws provided by the *Generalitat Valenciana* to support the public university system. These criteria take into account, for each degree course, concepts like the number of students, the number of academic years that the course exists, and so on.

Table 1 above shows the vectors of references and claims, together with the outcomes (in euros) using the compromise solutions. Note that the estate is greater than the total amount of the references, (404,816.17), and less than the total of claims, (4,078,097.61), so this is a *CERO* situation. Pulido et al. (2002) tackled this problem by first distributing the reference for each degree course and, afterwards, the remainder was shared according to a standard bankruptcy rule.

The last four columns represent the compromise solutions provided by the CEA and the proportional rule (which are very easy to implement) under the extreme (γ^f) and the diagonal ($\overline{\gamma}^f$) approaches. Note that the references have not to be respected. For example, using the proportional rule, under both approaches the third agent receives less than his reference amount. As indicated earlier, this is not the situation for the *CERO* approach described in Pulido et al. (2002), where each agent receives at least his reference.

References

Driessen, T., & Tijs, S. (1985). The τ-value, the core and semi-convex games. *International Journal of Game Theory*, *14*, 229–247.

O'Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences*, *2*, 345–371.

Pulido, M. (2001). *Estructuras de coaliciones y aplicaciones de la teoría de juegos*. Tesis Doctoral, Universidad Miguel Hernández, Elche, Spain.

Pulido, M., Sánchez-Soriano, J., & Llorca, N. (2002). Game theory techniques for university management: an extended bankruptcy model. *Annals of Operations Research*, *109*, 129–142.

Shapley, L. (1971). Cores of convex games. *International Journal of Game Theory*, *1*, 11–26.

Tijs, S. (1981). Bounds for the core and τ-value. In O. Moeschlin, & D. Pallaschke (Eds.) *Game theory and mathematical economics* (pp. 123–132). North-Holland, Amsterdam.