Improving Jacobi and Gauss-Seidel Iterations

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ABSTRACT

When convergent Jacobi or Gauss-Seidel iterations can be applied to solve systems of linear equations, a natural question is how convergence rates are affected if the original system is modified by performing some Gaussian elimination. We prove that if the initial iteration matrix is nonnegative, then such elimination improves convergence. Our results extend those contained in [4].

1. INTRODUCTION

Let us consider the system

$$x = Bx + b \tag{1.1}$$

where $B := (b_{ij})$, $b_{ij} \in \mathbb{R}$, $b_{ij} \geqslant 0$, for all i and j $(1 \leqslant i, j \leqslant n)$, $n \geqslant 2$, and $b \in \mathbb{R}$. In all that follows we shall also suppose that $b_{ii} = 0$ for all i, $1 \leqslant i \leqslant n$, and set $r(B) :=$ spectral radius of B.

We now fix k, $1 \leqslant k \leqslant n$, and consider the following system, obtained from (1.1) by elimination of x_k:

$$x = B'x + b'. \tag{1.2}$$

Here $B' := (b'_{ij})$ and $b' := (b'_i)$ are defined by

$$b'_{ij} := b_{ij} + b_{ik} b_{kj} \qquad \text{if} \quad j \neq k, \quad 1 \leqslant j \leqslant n,$$

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and

$$b'_{ik} := 0, \quad b'_i := b_i + b_{ik} b_k \qquad \text{if} \quad 1 \leqslant i \leqslant n.$$

REMARK 1.1. If we denote by S the matrix whose only nonvanishing terms belong to its kth column and coincide with the corresponding ones in B, and set $T := B - S$, then $B' = SB + T$ and $b' = (I + S)b$ (I denotes the identity matrix).

We can now obtain a relation between $r(B)$ and $r(B')$, which follows from Theorem 2 in [7] (see also §3 in [8]).

LEMMA 1.2. *One and only one of the following holds*:

(i) $r(B) = 0 = r(B')$,
(ii) $r(B) = 1 = r(B')$,
(iii) $0 < r(B)^2 \leqslant r(B') \leqslant r(B) < 1$,
(iv) $1 < r(B) \leqslant r(B') \leqslant r(B)^2$.

REMARK 1.3. It is not difficult to check that if $r(B) < 1$, then (1.1) and (1.2) are equivalent (see 2.3 in [6]).

2. ON JACOBI ITERATIONS

Lemma 1.2 and Remark 1.3 suggest that in case $r(B) < 1$, then Jacobi iterations will converge asymptotically faster to the solution of (1.1) when applied to (1.2) than when applied to the original system. Our aim now is to improve (iii) in Lemma 1.2 when B is irreducible. We denote by $\geqslant$, $\leqslant$ the order induced in $\mathbb{R}^n$ by the cone of vectors with nonnegative coordinates; we write $x < y$ if all the coordinates of y are greater than the corresponding ones of x. We shall also use the symbol $\leqslant$ for the ordering of matrices. We denote by B'_k the matrix of order $n-1$ obtained from B' by deleting its kth row and column.

LEMMA 2.1.

(i) *B' is reducible and $r(B') = r(B'_k)$.*
(ii) *If B is irreducible, then also B'_k is irreducible.*

Proof. (i): It is easy to exhibit a permutation matrix P such that

$$PB'P^{-1} = \begin{bmatrix} 0 & b_{k1} & \cdots & b_{k,k-1} & b_{k,k+1} & \cdots & b_{kn} \\ \vdots & & & B'_k & & & \\ 0 & & & & & & \end{bmatrix}.$$

(ii): This fact has already been mentioned in [2], and a proof for a particular case is given in [1]. A straightforward proof is obtained by noting that the strong connectedness of the graph of B'_k is inherited from that of the graph of B. ■

THEOREM 2.2. *If $r(B) < 1$ and B is irreducible, then $r(B') < r(B)$.*

Proof. Let a in $\mathbb{R}^n$ be such that $a > 0$ and $Ba = r(B)a$. Hence, if $i \neq k$,

$$\begin{aligned}\sum_{j \neq k} b_{ij}a_j + b_{ik}a_k &= \sum_{j \neq k} b_{ij}a_j + r(B)^{-1}b_{ik} \sum_{j \neq k} b_{kj}a_j \\ &= \sum_{j \neq k} \left(b_{ij} + r(B)^{-1}b_{ik}b_{kj}\right)a_j \geqslant \sum_{j \neq k} b'_{ij}a_j.\end{aligned}$$

Thus if we define a' in $\mathbb{R}^{n-1}$ by $a'_i := a_i$ for $i \leqslant k-1$, and $a'_i := a_{i+1}$ when $k \leqslant i \leqslant n-1$, the inequalities above yield

$$B'_k a' \leqslant r(B)a'. \tag{2.1}$$

Since $a' > 0$, Theorem 2.2 in [9] implies that $r(B'_k) \leqslant r(B)$. But the irreducibility of B implies that $b_{ik}b_{kj} \neq 0$ for some $i \neq k \neq j$. Thus, equality does not hold in (2.1); this fact and Theorem 2.2 in [9] now imply that $r(B'_k) < r(B)$. ■

LEMMA 2.3. *If $r(B) < 1$, then $c_{ii} := \sum_j b_{ij}b_{ji} < 1$ for all i, $1 \leqslant i \leqslant n$.*

Proof. Consider the diagonal matrix $C := (c_{ii})$; since $0 \leqslant C \leqslant B^2$, we get $\max\{c_{ii}\} = r(C) \leqslant r(B^2) = r(B)^2 < 1$. ■

COROLLARY 2.4. *If $r(B) < 1$ and $i \neq j$, then $b_{ij}b_{ji} < 1$. Thus, it is possible to eliminate the diagonal terms in (1.2), yielding the following* *equivalent system*:

$$x = B''x + b'' \tag{2.2}$$

with

$$b''_{ij} := \frac{b'_{ij}}{1 - b_{ik}b_{ki}} \quad \textit{if} \quad i \neq j \neq k,$$

$$b''_{ii} := 0, \qquad b''_{ik} := 0, \qquad b''_{i} := \frac{b'_{i}}{1 - b_{ik}b_{ki}} \quad \textit{for } 1 \leqslant i \leqslant n.$$

In [4], the possibility of getting (2.2) was based on the following hypotheses made on B (besides $0 \leqslant B$):

(i) $\sum_{1 \leqslant j \leqslant n} b_{ij} \leqslant 1$ for all i, $1 \leqslant i \leqslant n$.
(ii) Strict inequality holds for at least one i in (i).
(iii) $I - B$ is invertible.

Note that (ii) follows from (i) and (iii). Now, because of Gerschgorin's theorem, (i) implies that $r(B) \leqslant 1$; thus, the Perron-Frobenius theorem and (iii) imply that $r(B) < 1$. The following simple example shows that $r(B) < 1$ does not imply (i). Take

$$B := \begin{bmatrix} 0 & 0 & 1.5 \\ 0.5 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix};$$

Then

$$r(B) = (0.75)^{1/3}.$$

The discretization of linear elliptic problems with nonconstant coefficients may often lead to situations similar to the one described in the example above. In what follows we obtain results that improve those contained in [4]: we assume from now on that $r(B) < 1$. Let B''_k denote the principal submatrix obtained from B'' by deleting its kth row and column.

THEOREM 2.5.

(i) $r(B'') = r(B''_k)$.

(ii) *If B is irreducible and there exists i such that $b_{ik}b_{ki} \neq 0$, then $r(B''_k) < r(B'_k)$; otherwise $r(B''_k) \leqslant r(B'_k)$.*

(iii) *If $b_{ik}b_{ki} \neq 0$ for all $i \neq k$, $1 \leqslant i \leqslant n$, then $r(B''_k) < r(B'_k)$.*

Proof. (i): This part can be dealt with like Lemma 2.1(i).

(ii): Note that, if we set $D_k :=$ diagonal matrix of B_k', then

$$B_k'' = (I - D_k)^{-1}(B_k' - D_k). \tag{2.3}$$

Since B_k' is irreducible, $r(B_k' - D_k) < r(B_k')$ if and only if $D_k \neq 0$, and in this case, $r(B_k'') < r(B_k')$ follows from (2.3) and the Stein-Rosenberg theorem as stated in [5].

Consider now the general case $0 \leqslant B$. For positive real t, we define $B(t)$ by $b_{ij}(t) := b_{ij} + t$, for $i \neq j$ and $b_{ii}(t) := 0$. There exists $t_0 > 0$ such that if $0 < t < t_0$ then $r(B(t)) < 1$. For any such t, we define $B_k'(t)$ and $B_k''(t)$, starting from $B(t)$, in the same way we defined B_k' and B_k'' starting from B. Thus, $r(B_k''(t)) < r(B_k'(t))$, and letting t tend to 0, we obtain the conclusion. ∎

COROLLARY 2.6. *If B is irreducible, then $r(B'') < r(B)$. If B is also symmetric, then $r(B'') < r(B') < r(B)$.*

3. ON GAUSS-SEIDEL ITERATIONS

We want to establish some facts that relate the convergence rate of Gauss-Seidel iterations for (1.1) and (1.2); we base our discussion on the extended version of the Stein-Rosenberg theorem given in [5]. If S and T are square nonnegative matrices, 1.8 in [5] easily implies that if $r(S) < r(S+T)$, then the function $r(S+tT)$, t in $\mathbb{R}$, $t \geqslant 0$, is unbounded; moreover, if $r(S) < 1$, then the unique $t_1 > 0$ such that $r(S + t_1 T) = 1$ (see 1.7 in [5]) also satisfies $r((I-S)^{-1}T) = t_1^{-1}$.

In the sequel L and U will denote the lower and upper part matrices of B; analogously, L' is the strict lower part matrix of B' and $U' := B' - L'$. The Gauss-Seidel matrices associated to these splittings of B and B' are, respectively, $H := (I-L)^{-1}U$ and $H' := (I-L')^{-1}U'$.

The results quoted above from [5] imply that if $r(B') > 0$, then $r(H') > 0$ and there exists a unique $t_1' > 0$ such that $r(L' + t_1'U') = r(L_k' + t_1'U_k') = 1$ and $r(H') = (t_1')^{-1}$.

THEOREM 3.1. *If B is irreducible and if there exist i and j such that $b_{ik}b_{kj} \neq 0$, with* (a) $k < j < i$, (b) $j < i < k$, *or* (c) $i < k < j$, *then $r(H') < r(H)$.*

Proof. Consider t_1 as above and x in $\mathbb{R}^n$, $x > 0$, such that

$$x = (L + t_1 U)x.$$

For any i, if $k < i$, we get

$$\begin{aligned} x_i &= \sum_{\substack{s<i\\ s\neq k}} b_{is}x_s + b_{ik}\Big(\sum_{m<k} b_{km}x_m + t_1 \sum_{m>k} b_{km}x_m\Big) + t_1 \sum_{s>i} b_{is}x_s \\ &= \sum_{s<k} (b_{is} + b_{ik}b_{ks})x_s + \sum_{k<s<i} (b_{is} + t_1 b_{ik}b_{ks})x_s \\ &\quad + t_1\Big(\sum_{s\geqslant i} (b_{is} + b_{ik}b_{ks})x_s\Big). \end{aligned} \tag{3.1}$$

If now $i < k$,

$$\begin{aligned} x_i &= \sum_{s<i} b_{is}x_s + t_1 b_{ik}\Big(\sum_{m<k} b_{km}x_m + t_1 \sum_{m>k} b_{km}x_m\Big) + t_1 \sum_{\substack{s>i\\ s\neq k}} b_{is}x_s \\ &= \sum_{s<i} (b_{is} + t_1 b_{ik}b_{ks})x_s + t_1 \sum_{i\leqslant s<k} (b_{ik}b_{ks} + b_{is})x_s \\ &\quad + t_1 \sum_{s>k} (b_{is} + t_1 b_{ik}b_{ks})x_s. \end{aligned} \tag{3.2}$$

Thus

$$x_i \geqslant \sum_{\substack{s<i\\ s\neq k}} (b_{is} + b_{ik}b_{ks})x_s + t_1 \sum_{\substack{s\geqslant i\\ s\neq k}} (b_{is} + b_{ik}b_{ks})x_s \tag{3.3}$$

for all $i \neq k$, $1 \leqslant i \leqslant n$.

If (a) holds, then, for the corresponding i, (3.1) implies the strict inequality in (3.3). Analogously, when either (b) or (c) holds, then (3.2) implies that the inequality (3.3) is strict for some i. Thus, if we denote by x' the vector obtained from x by deleting x_k, the hypotheses imply that $(L'_k + t_1 U'_k)x' \leqslant x'$, with equality excluded.

Since B'_k is irreducible, we have that $r(L'_k + t_1 U'_k) < 1$ (see Theorem 2.2 in [9]). Hence, if $t'_1 > 0$ is such that $r(L'_k + t'_1 U'_k) = 1$, it must satisfy $t_1 < t'_1$, which yields

$$r(H') = r\big((I - L'_k)^{-1} U'_k\big) = (t'_1)^{-1} < t_1^{-1} = r(H). \qquad \blacksquare$$

COROLLARY 3.2. $r(H') \leqslant r(H)$.

Proof. A standard limit argument, similar to the one given in the proof of (ii) in Theorem 2.5, implies the conclusion. ■

If L'' and U'' denote the lower and upper parts of B'', we set $H'' := (I - L'')^{-1}U''$. The following result can be proven by reasoning as in the proof of Theorem 3.1.

THEOREM 3.3. *If B is irreducible and if there exists i such that $b_{ik}b_{ki} \neq 0$, then $r(H'') < r(H')$. Otherwise $r(H'') \leqslant r(H')$.*

COROLLARY 3.4. *If B is symmetric and irreducible, then $r(H'') < r(H')$.*

EXAMPLE 3.5.

(i) Let us consider the primitive matrix

$$B := \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ d & 0 & 0 \end{bmatrix},$$

with a, b, c, and d positive real numbers such that $r(B) < 1$. In any of the cases $k := 1$, $k := 2$, or $k := 3$, Theorem 3.1 can be applied and we get that

$$r(H') = d(b + ac) < r(H).$$

(ii) If we now interchange the second and third coordinates, i.e. if we define

$$B := \begin{bmatrix} 0 & b & a \\ d & 0 & 0 \\ 0 & c & 0 \end{bmatrix},$$

then for each k we have

$$r(H') = r(H) = d(b + ac).$$

Thus, the sufficient conditions in Theorem 3.1 are also necessary in a general setting in order to have $r(H') < r(H)$.

EXAMPLE 3.6.

(i) If we now let $b := 0$ in Example 3.5(i) (B turns to be cyclic), we get, for any k, $r(H') = acd < r(H) = (acd)^{1/2}$.

(ii) On the other hand, with $b := 0$ in Example 3.5(ii), we get $r(H') = acd = r(H)$.

These simple examples show that a better ordering of the unknowns in order to apply Gauss-Seidel iterations may not be a better one when such iterations will be applied after elimination. Examples 3.5(ii) and 3.6(ii) also give evidence that irreducibility does not guarantee improvement in Gauss-Seidel iterations after elimination.

4. SOME HEURISTICS

A reasonable question concerning the elimination of nodes is whether an optimal choice of k in (1.2) can be made in such a way that the corresponding B' has minimal spectral radius [when $r(B) < 1$]; furthermore, whether there is a heuristic argument that says a good choice of k in obtaining B' is where some norm of the kth column of B is maximal. A negative answer to the latter question easily follows from the next two lemmas; their proof is straightforward.

We set

$$B := \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ d & 0 & 0 \end{bmatrix}$$

with a, b, c, and d positive real numbers.

LEMMA 4.1. *With B as above, the following hold:*

(i) $r(B) < 1$ *if and only if* $bd + acd < 1$.
(ii) $r(B) = 1$ *if and only if* $bd + acd = 1$.

LEMMA 4.2. *Let us denote by B_1, B_2, and B_3 the matrices B' obtained in* (1.2) *by taking $k = 1$, 2, or 3, respectively. Then the following hold:*

(i) $r(B_1) = r(B_3) < r(B_2)$ *if and only if* $r(B) < 1$.
(ii) $r(B_1) = r(B_3) > r(B_2)$ *if and only if* $r(B) > 1$.

Note that in the graph of B there are three edges involving the first and third nodes, but there are only two involving the second one. Thus, a natural guess is that for elimination, the number of edges involving a node, i.e. its degree, is more relevant than their sizes.

In experiments with matrices that arise in the discretization of linear elliptic problems with constant coefficients, we have obtained that the farther in the eliminated node, the faster the convergence of Jacobi iterations; we have no proof for a general statement. If the coefficients of the elliptic problem are variable, the sizes of the edges affecting a node do play a role. To show this, consider first

$$B := \begin{bmatrix} 0 & a & 0 \\ a & 0 & b \\ 0 & b & 0 \end{bmatrix},$$

with a and b positive real numbers. We clearly have $r(B) = (a^2 + b^2)^{1/2}$. In what follows, we adopt the notation introduced in Lemma 4.2.

LEMMA 4.3. *The following hold:*

(i) $r(B_2) < r(B_1)$ *and* $r(B_2) < r(B_3)$ *if and only if* $r(B) < 1$.
(ii) $r(B_2) > r(B_1)$ *and* $r(B_2) > r(B_3)$ *if and only if* $r(B) > 1$.

Note that $r(B_2) = a^2 + b^2$. Consider now

$$B := \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ a & 0 & b & 0 & 0 \\ 0 & b & 0 & c & 0 \\ 0 & 0 & c & 0 & d \\ 0 & 0 & 0 & d & 0 \end{bmatrix},$$

with a, b, and d positive real numbers such that $d < a^2 + b^2$. It follows from Lemma 4.3(i) that there exists $c_0 > 0$ such that if $0 \leqslant c < c_0$ then $r(B_2) < r(B_3)$. Thus it seems that a nearly optimal choice in the elimination of a node might be achieved by taking account of its degree, the size of the edges involving it and, last but not least, its depth. Hence, any good strategy for partial elimination should be based on a thorough study of the graph of the matrix.

So far, we have taken a purely analytic point of view. On the other hand, if we are interested in the computational aspects of elimination and how it affects the complexity of the problems to be treated, good analytic strategies can turn to be computationally disastrous. To illustrate this, consider a matrix

B with a star graph (see [3]); the elimination of the inner node produces full fill-in.

We shall examine the relations of our results with preconditioning methods in another paper.

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