



NORTH-HOLLAND

## Improving the Modified Gauss-Seidel Method for Z-Matrices

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### ABSTRACT

In 1991 A. D. Gunawardena et al. reported that the convergence rate of the Gauss-Seidel method with a preconditioning matrix  $I + S$  is superior to that of the basic iterative method. In this paper, we use the preconditioning matrix  $I + S(\alpha)$ . If a coefficient matrix  $A$  is an irreducibly diagonally dominant Z-matrix, then  $[I + S(\alpha)]A$  is also a strictly diagonally dominant Z-matrix. It is shown that the proposed method is also superior to other iterative methods. © 1997 Elsevier Science Inc.

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### 1. INTRODUCTION

Let us consider iterative methods for the solution of the linear system

$$Ax = b, \tag{1}$$

where  $A$  is an  $n \times n$  square matrix, and  $x$  and  $b$  are vectors. Then the basic iterative scheme for Equation (1) is

$$Mx_{k+1} = Nx_k + b, \quad k = 0, 1, \dots, \tag{2}$$

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where  $A = M - N$ , and  $M$  is nonsingular. Thus (2) can also be written as

$$x_{k+1} = Tx_k + c, \quad k = 0, 1, \dots, \tag{3}$$

where  $T = M^{-1}N$ ,  $c = M^{-1}b$ . Assuming  $A = I - L - U$ , where  $I$  is the identity matrix, and  $L$  and  $U$  are strictly lower and strictly upper triangular matrices, respectively, the iteration matrix of the classical Gauss-Seidel method is given by  $T = (I - L)^{-1}U$ .

We now transform the original system (1) into the preconditioned form

$$PAx = Pb. \tag{4}$$

Then, we can define the basic iterative scheme:

$$M_p x_{k+1} = N_p x_k + Pb, \quad k = 0, 1, \dots, \tag{5}$$

where  $PA = M_p - N_p$  and  $M_p$  is nonsingular.

Recently, Gunawardena et al. [1] proposed the modified Gauss-Seidel method with  $P = I + S$ , where

$$S = \begin{pmatrix} 0 & -a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1n} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The performance of this method on some matrices is investigated in [1].

In this paper, we propose a scheme for improving of the modified Gauss-Seidel method and discuss convergence. Finally, we show that this method yields a considerable improvement in the rate of convergence.

## 2. PROPOSED METHOD

First, let us summarize the modified Gauss-Seidel method [1] with the preconditioner  $P = I + S$ . Let all elements  $a_{i+1}$  of  $S$  be nonzero. Then we have

$$\begin{aligned} \tilde{A}x &= (I + S)Ax = [I - L - SL - (U - S + SU)]x, \\ \tilde{b} &= (I + S)b. \end{aligned} \tag{6}$$

Whenever

$$a_{ii+1}a_{i+1i} \neq 1 \quad \text{for } i = 1, 2, \dots, n - 1,$$

$(I - SL - L)^{-1}$  exists, and hence it is possible to define the Gauss-Seidel iteration matrix for  $\tilde{A}$ , namely

$$\tilde{T} = (I - SL - L)^{-1}(U - S + SU). \tag{7}$$

This iteration matrix  $\tilde{T}$  is called the modified Gauss-Seidel iteration matrix.

We next propose a new iterative method with the preconditioned matrix,

$$P = I + S(\alpha),$$

where  $S(\alpha)$  is

$$S(\alpha) = \begin{pmatrix} 0 & -\alpha_1 a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -\alpha_2 a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1} a_{n-1n} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we obtain

$$A(\alpha) = [I + S(\alpha)]A = I - L - S(\alpha)L - [U - S(\alpha) + S(\alpha)U], \tag{8}$$

$$b(\alpha) = [I + S(\alpha)]b.$$

Whenever

$$\alpha_i a_{ii+1} a_{i+1i} \neq 1 \quad \text{for } i = 1, 2, \dots, n - 1,$$

$[I - S(\alpha)L - L]^{-1}$  exists, and hence it is possible to define the Gauss-Seidel iteration matrix for  $A(\alpha)$ , namely

$$T(\alpha) = [I - S(\alpha)L - L]^{-1}[U - S(\alpha) + S(\alpha)U]. \tag{9}$$

REMARK 1. In (9), if  $\alpha_i = 0$  for all  $i$ ,  $T(\alpha)$  reduces to the classical Gauss-Seidel iterative method, and if  $\alpha_i = 1$  for all  $i$ ,  $T(\alpha)$  reduces to the modified Gauss-Seidel iterative method.

## 3. CONVERGENCE OF THE PROPOSED METHOD

First, we give a well-known result [2, 3].

LEMMA 2. *An upper bound on the spectral radius  $\rho(T)$  for the Gauss-Seidel iteration matrix  $T$  is given by*

$$\rho(T) \leq \max_i \frac{\bar{u}_i}{1 - \bar{l}_i},$$

where  $\bar{l}_i$  and  $\bar{u}_i$  are the sums of the moduli of the elements in row  $i$  of the triangular matrices  $L$  and  $U$ , respectively.

Next, we discuss the convergence of the proposed method. Let  $A(\alpha) = D(\alpha) - E(\alpha) - F(\alpha)$ , where  $D(\alpha)$ ,  $-E(\alpha)$ , and  $-F(\alpha)$  are the diagonal, strictly lower triangular, and strictly upper triangular parts of  $A(\alpha)$ . Then the elements of  $A(\alpha)$  are

$$\bar{a}_{ij} = \begin{cases} a_{ij} - \alpha_i a_{i+1} a_{i+1j}, & 1 \leq i < n, \\ a_{nj}. & i = n \end{cases} \quad (10)$$

If  $A$  is a diagonally dominant  $Z$ -matrix, then we have

$$\begin{aligned} 0 \leq a_{i+1} a_{i+1j} \leq 1 & \quad \text{for } j \neq i + 1, \\ -1 \leq a_{i+1} a_{i+1i+1} \leq 0. \end{aligned} \quad (11)$$

Therefore, the following inequalities hold:

$$\begin{aligned} a_{i+1} a_{i+1i} & \geq 0 \\ a_{i+1} \sum_{j=1}^{i-1} a_{i+1j} & \geq 0, \\ a_{i+1} \sum_{j=i+1}^n a_{i+1j} & \leq 0, \quad 1 \leq i < n. \end{aligned}$$

For simplicity we denote

$$\begin{aligned}
 p_i &= a_{i+1}a_{i+1i}, \\
 q_i &= a_{i+1} \sum_{j=1}^{i-1} a_{i+1j}, \\
 r_i &= a_{i+1} \sum_{j=i+1}^n a_{i+1j}, \quad \text{for } 1 \leq i < n,
 \end{aligned}$$

and set

$$\begin{aligned}
 p_n &= 0, \\
 q_n &= 0, \\
 r_n &= 0.
 \end{aligned}$$

Then the following inequality holds:

$$p_i + q_i + r_i = a_{i+1} \sum_{j=1}^n a_{i+1j} \leq 0, \quad 1 \leq i < n.$$

Furthermore, if  $a_{i+1} \neq 0$  and  $\sum_{j=1}^n a_{i+1j} < 0$  for some  $i < n$ , then we have

$$p_i + q_i + r_i < 0 \quad \text{for some } i < n. \tag{12}$$

**THEOREM 3.** *Let  $A$  be a nonsingular diagonally dominant Z-matrix with unit diagonal elements and  $\sum_{j=1}^n a_{nj} > 0$ . Assume that  $\sum_{j=1}^n a_{i+1j} > 0$  if  $\sum_{j=1}^n a_{ij} = 0$  for some  $i < n$ . Then  $A(\alpha)$  is a strictly diagonally dominant Z-matrix, and  $\rho(T(\alpha)) < 1$  for  $0 \leq \alpha_i \leq 1$  ( $1 \leq i < n$ ).*

*Proof.* Let  $d(\alpha)_i$ ,  $l(\alpha)_i$ , and  $u(\alpha)_i$  be the sums of the elements in row  $i$  of  $D(\alpha)$ ,  $L(\alpha)$ , and  $U(\alpha)$ , respectively. The following equations hold:

$$\begin{aligned}
 d(\alpha)_i &= \bar{a}_{ij} = 1 - \alpha_i p_i, \quad 1 \leq i \leq n, \\
 l(\alpha)_i &= - \sum_{j=1}^{i-1} \{\bar{a}_{ij}\} = l_i + \alpha_i q_i, \quad 1 \leq i \leq n, \\
 u(\alpha)_i &= - \sum_{j=i+1}^n \{\bar{a}_{ij}\} = u_i + \alpha_i r_i, \quad 1 \leq i \leq n,
 \end{aligned} \tag{13}$$

where  $l_i$  and  $u_i$  are the sums of the elements in row  $i$  of  $L$  and  $U$  for  $A = I - L - U$ , respectively. Since  $A$  is a diagonally dominant  $Z$ -matrix, from (11) the following relations hold:

$$1 - \alpha_i a_{i+1} a_{i+1j} > 0 \quad \text{for } j = i,$$

$$a_{ij} - \alpha_i a_{i+1} \sum_{k=1}^{i-1} a_{i+1k} \leq 0 \quad \text{for } i > j,$$

$$(1 - \alpha_i) a_{ij} - \alpha_i a_{i+1} \sum_{k=i+2}^n a_{i+1k} \leq 0 \quad \text{for } i < j.$$

Therefore,  $l(\alpha)_i \geq 0$ ,  $u(\alpha)_i \geq 0$ , and  $A(\alpha)$  is a  $Z$ -matrix. Moreover, from (12) and the assumption, we can easily obtain

$$d(\alpha)_i - l(\alpha)_i - u(\alpha)_i = (d_i - l_i - u_i) - \alpha_i(p_i + q_i + r_i) > 0$$

for all  $i$ . (14)

Therefore,  $A(\alpha)$  satisfies the condition of diagonal dominance. From  $u(\alpha)_i \geq 0$ , we have

$$d(\alpha)_i - l(\alpha)_i > u(\alpha)_i \geq 0 \quad \text{for all } i.$$

This implies

$$\frac{u(\alpha)_i}{d(\alpha)_i - l(\alpha)_i} < 1. \quad (15)$$

Hence,  $\rho(T(\alpha)) < 1$ , by Lemma 3. ■

**THEOREM 4.** *Let  $A$  be a matrix satisfying the conditions in Theorem 3. Put  $\alpha'_i = (1 - l_i - u_i - 2a_{i+1}) / (p_i + q_i + r_i - 2a_{i+1})$  for  $1 \leq i < n$ . Then  $\alpha'_i > 1$ ,  $A(\alpha)$  is a strictly diagonally dominant matrix, and  $\rho(T(\alpha)) < 1$  for  $1 \leq \alpha_i < \alpha'_i$ .*

*Proof.* Since  $\sum_{j=1, j \neq i}^n a_{i+1j} \leq 0$ , we have

$$\begin{aligned}
 p_i + q_i + r_i - 2a_{i+1} &= a_{i+1} \left( \sum_{j=1}^n a_{i+1j} - 2 \right) \\
 &= a_{i+1} \left( \sum_{\substack{j=1 \\ j \neq i+1}}^n a_{i+1j} - 1 \right) > 0 \quad \text{for } 1 \leq i < n,
 \end{aligned}
 \tag{16}$$

and

$$1 - l_i - u_i - 2a_{i+1} > p_i + q_i + r_i - 2a_{i+1} > 0 \quad \text{for } 1 \leq i < n,$$

since  $p_i + q_i + r_i < 0$ . This implies

$$\frac{1 - l_i - u_i - 2a_{i+1}}{p_i + q_i + r_i - 2a_{i+1}} > 1 \quad \text{for } 1 \leq i < n.$$

That is,  $\alpha'_i > 1$  for  $1 \leq i < n$ . Let

$$\bar{u}(\alpha)_i = \sum_{j=i+1}^n |a_{ij} - \alpha_i a_{i+1} a_{i+1j}| \quad \text{for } 1 \leq i < n.$$

Then for  $\alpha_i > 1$  ( $1 \leq i < n$ ) the following relation holds:

$$\begin{aligned}
 \bar{u}(\alpha)_i &= |(1 - \alpha_i) a_{i+1}| + \sum_{j=i+2}^n |a_{ij} - \alpha_i a_{i+1} a_{i+1j}| \\
 &= (1 - \alpha_i) a_{i+1} - \sum_{j=i+2}^n (a_{ij} - \alpha_i a_{i+1} a_{i+1j}) \\
 &= 2(1 - \alpha_i) a_{i+1} - \sum_{j=i+1}^n (a_{ij} - \alpha_i a_{i+1} a_{i+1j}) \\
 &= (u_i + 2a_{i+1}) + \alpha_i (r_i - 2a_{i+1}) \geq 0.
 \end{aligned}
 \tag{17}$$

Thus from (13) and (17) we easily obtain for  $1 \leq \alpha_i < \alpha'_i$  ( $1 \leq i < n$ )

$$\begin{aligned} d(\alpha)_i - l(\alpha)_i - \bar{u}(\alpha)_i &= (1 - l_i) - \alpha_i(p_i + q_i) - (u_i + 2a_{i,i+1}) - \alpha_i(r_i - 2a_{i,i+1}) \\ &= (1 - l_i - u_i - 2a_{i,i+1}) - \alpha_i(p_i + q_i + r_i - 2a_{i,i+1}) > 0. \end{aligned}$$

Therefore,  $A(\alpha)$  is a strictly diagonally dominant matrix, and thus the following equality holds:

$$\rho(T(\alpha)) \leq \frac{\bar{u}(\alpha)_i}{d(\alpha)_i - l(\alpha)_i} < 1 \quad \text{for } 1 \leq \alpha_i \leq \alpha'_i \quad (1 \leq i \leq n).$$

Hence, an application of Lemma 2 yields  $\rho(T(\alpha)) < 1$  for  $1 \leq \alpha_i < \alpha'_i$  ( $1 \leq i < n$ ). ■

The behavior of the spectral radius of the proposed method as a function of  $\alpha_i = \alpha$  for the strictly diagonally dominant  $Z$ -matrix  $A$  is shown in Fig. 1.

The variation of the spectral radius of the proposed method is extremely small compared with that of the SOR method, as shown in Figure 1.

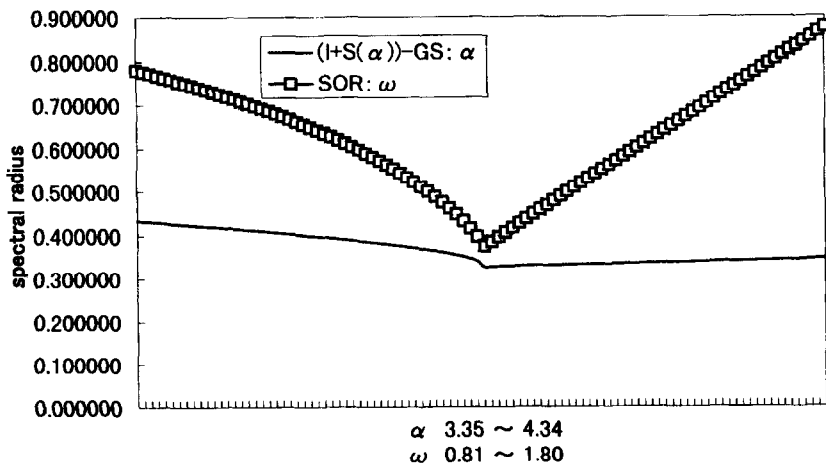


FIG. 1. The spectral radii of the proposed method and the SOR method for  $n = 10$ .



Moreover, our convergence curve is relatively flat for  $\alpha > \alpha_{\text{opt}}$ . However, it is extremely difficult to compute an optimal  $\alpha_i$  directly from Theorem 4. Therefore we propose a practical technique for its determination.

To find  $\alpha_i$ , we dictate that the equality holds in (17):

$$(u_i + 2a_{ii+1}) + \alpha_i(r_i - 2a_{ii+1}) = 0, \quad 1 \leq i < n.$$

Solving this equation, we have

$$\alpha_i = \frac{u_i + 2a_{ii+1}}{2a_{ii+1} - r_i}, \quad 1 \leq i < n \tag{18}$$

#### 4. NUMERICAL EXAMPLES AND CONCLUSION

We now test the validity of the determination (18). To do so, we consider the following matrix:

$$A = \begin{pmatrix} 1 & c_1 & c_2 & c_3 & c_1 & \cdots \\ c_3 & 1 & c_1 & c_2 & \ddots & c_1 \\ c_2 & c_3 & \ddots & \ddots & \ddots & c_3 \\ c_1 & \ddots & \ddots & 1 & c_1 & c_2 \\ c_3 & \ddots & c_2 & c_3 & 1 & c_1 \\ \cdots & c_3 & c_1 & c_2 & c_3 & 1 \end{pmatrix},$$

where  $c_1 = -1/n$ ,  $c_2 = -1/(n + 1)$ , and  $c_3 = -1/(n + 2)$ . We set  $b$  [see (1)] such that the solution is  $x^T = (1, 2, \dots, n)$ . Let the convergence criterion be  $\|x^{k+1} - x^k\|/\|x^{k+1}\| \leq 10^{-6}$ . We show CPU times and the number of iterations in Table 1 for  $n = 20, 30, 50$ , and  $100$ . For comparison, we also show results for unpreconditioning (GS), the modified Gauss-Seidel method (MGS) [1], and the adaptive Gauss-Seidel method (AGS) [4].

The iteration number for the proposed method is larger than that for AGS [4], while the CPU time for the proposed method is smaller than that for AGS. An optimum parameter  $\omega_{\text{opt}}$  of the SOR method was obtained by numerical computation. We also obtained the optimum parameter  $\alpha_{\text{opt}}$  of the proposed method by replacing  $\alpha$  with  $\alpha_i$  ( $i = 1, 2, \dots, n - 1$ ) by numerical computation.

TABLE 1  
ITERATION NUMBERS AND CPU TIMES FOR A Z-MATRIX

$n$	Proposed method																	
	Optimum			Determine			GS			MCS			AGS			SOR <sub>opt</sub>		
	No. of iters.	$\alpha_{opt}$	Time (s)	No. of iters.	Time (s)	Time (s)	No. of iters.	Time (s)	Time (s)	No. of iters.	Time (s)	Time (s)	No. of iters.	Time (s)	Time (s)	No. of iters.	$\omega_{opt}$	Time (s)
20	19	10.4	0.01	31	0.01	0.01	65	0.02	59	0.02	35	0.03	20	1.50	0.01			
30	23	17.4	0.01	48	0.02	0.02	93	0.06	87	0.05	50	0.09	25	1.55	0.01			
50	28	32.3	0.04	80	0.10	0.10	146	0.16	141	0.15	79	0.33	30	1.65	0.03			
100	38	72.9	0.22	156	0.62	0.62	269	1.01	265	1.00	148	2.56	42	1.75	0.18			

TABLE 2  
MODEL PROBLEM

$m$	Proposed method																	
	Optimum			Determine			GS			MCS			AGS			SOR <sub>opt</sub>		
	No. of iters.	$\alpha_{opt}$	Time (s)	No. of iters.	Time (s)	Time (s)	No. of iters.	Time (s)	Time (s)	No. of iters.	Time (s)	Time (s)	No. of iters.	Time (s)	Time (s)	No. of iters.	$\omega_{opt}$	Time (s)
10	20	2.65	0.06	20	0.06	0.06	110	0.27	69	0.18	50	0.56	24	1.53	0.10			
15	26	3.0	0.61	45	1.01	1.01	230	4.85	144	3.00	108	7.74	34	1.66	0.73			
20	34	3.2	2.58	82	6.5	6.5	385	29.24	242	18.7	185	54.65	54	1.73	4.15			

Finally, we consider a model problem [5, p. 202]. We use a standard central-difference formula and a uniform mesh with length  $h = 1/m$ . Table 2 shows CPU times and the number of iterations for the model problem. We adopt the theoretical value

$$\omega_{\text{opt}} = \frac{2}{1 + \sin(\pi/m)}$$

for the SOR method.

In this paper, we have proposed a new algorithm based on the Gauss-Seidel method. As a result we have succeeded in improving the convergence of this method. We have shown that the spectral radius of the proposed method with  $\alpha_{\text{opt}}$  is smaller than that of the SOR method.

#### REFERENCES

- 1 A. D. Gunawardena, S. K. Jain, and L. Snyder, Modified iterative methods for consistent linear systems, *Linear Algebra Appl.* 154-156:123-143 (1991).
- 2 K. R. James, Convergence of matrix iterations subject to diagonal dominance, *SIAM J. Numer. Anal.* 10:478-484 (1973).
- 3 K. R. James and W. Riha, Convergence criteria for successive overrelaxation, *SIAM J. Numer. Anal.* 12(2):137-143 (Apr. 1975).
- 4 M. Usui, H. Niki, and T. Kohno, Adaptive Gauss-Seidel method for linear systems, *Internatl. J. Comput. Math.* 51:119-125 (1994).
- 5 R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Engelwood Cliffs, N.J., 1962.

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