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Transformation of optimization problems in revenue management, queueing system, and supply chain management

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ABSTRACT

For revenue optimization problems in the literature on revenue management, supply chain management, and queueing systems, some assumptions (such as concavity of revenue functions or increasing generalized failure rate) are often needed to ensure the problems to be analytically tractable. We show that these assumptions are not necessary. For this, we present and study a parametric revenue maximization problem to unify some problems in the literature. Without the usual assumptions, we transform the problem into an equivalent one where the revenue function is increasing, continuous and concave. We then apply the transformation method to a continuous time revenue management problem and conclude that the monotone results are robust to demand function and allowable price set. Also, we apply the transformation method to study a parametric cost minimization problem. We further apply our method to two optimal control problems in queueing systems and an inventory control problem in a supply chain with price-only contract.

1. Introduction

Revenue maximization problems appear frequently in the literature, typically those on revenue management (Gallego and van Ryzin, 1994), optimality of queueing systems (Lippman, 1975) and supply chain management (Lariviere and Porteus, 2001). To ensure these problems to be analytically tractable there often needs some assumptions. Ziya et al. (2004) summarize and discuss three famous assumptions presented in the literature. The first two are the concavity of the revenue function with demand and price, respectively, and the third is the increasing generalized failure rate (IGFR) of the demand distribution function under which the revenue function is unimodal. Ziya et al. (2004) show that none of these assumptions implies any other. These assumptions appear in papers concerning revenue management, inventory and pricing in supply chain management, network services, auction and mechanism design, and price competition (Ziya et al., 2004). However, we no longer need these assumptions, as shown in this paper.

We first present a parametric revenue maximization problem to unify several revenue maximization problems discussed in the literature. Without the assumptions presented in the literature, we transform the problem into an equivalent well structured one in which the revenue function is increasing, continuous and concave. Thus, the resulting maximization problem is analytically tractable. The transformation here is algorithmic. We illustrate the problem and the results by an optimal arrival control in queueing systems, which is not concerned in Ziya et al. (2004).

We then apply the transformation to study the continuous time revenue management. Revenue management deals with pricing and allocation problems in many industries of selling fixed stock items over a finite horizon by controlling price. These industries include airlines selling seats before planes' departing, hotels' renting rooms before midnight, and retailers' selling seasonal items with long procurement lead time. The study on revenue management dates back to Littlewood (1972) for a stochastic two-fare and single-leg problem in the airlines. Li (1988) presents a continuous time model with demand of a controlled Poisson process. Gallego and van Ryzin (1994) study continuous time revenue management, where demands (customers) arrive according to a homogeneous Poisson process with price related demand rate, and price is chosen from the set $[0,\infty)$. They assume a *regular demand function*, that is, the corresponding revenue function (i.e., the demand rate times price) is a continuous, bounded and concave function of the demand rate, and tends to zero as the demand rate tends to zero. With the regular demand function, they show monotonicity and concavity of the optimal expected revenue and monotonicity of the optimal pricing policy.

The work of Gallego and van Ryzin (1994) has been extended into several directions: (1) to relax the assumption of the regular demand, for example, in Zhao and Zheng (2000) and Wei and

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Hu (2002); (2) to extend the allowable price set to a discrete set, for example, in Chatwin (2000), Feng and Xiao (2000a, 2000b), and Feng and Gallego (2000); (3) to study the revenue management problems in network environments, for example, in Ge et al. (2010), Dai et al. (2005), Graf and Kimms (2013); (4) to study the multi-period revenue management, for example, in Talluri and van Ryzin (2004) and Du et al. (2005); and (5) to study the revenue management in competitive environments, for example, in Netessine and Shumsky (2005), Hu et al. (2010), Huang et al. (2013), and Wei et al. (2013).

We apply the transformation to study a continuous time revenue management problem along the first and second directions pointed above. With a general demand function (that may be neither decreasing nor concave) and an arbitrary allowable price set (that can be, e.g., an interval, a discrete set, or even combination of intervals and discrete points), we show that the problem can be transformed into an equivalent one, where the revenue function is continuous and concave (i.e., the corresponding demand function is regular) and increasing. Thus, directly citing the results in the literature, e.g., Gallego and van Ryzin (1994) and Wei and Hu (2002), we get the usual monotone properties of the optimal policies and the concavity of the optimal value function. Hence, these monotone properties are robust to demand function and allowable price set.

The supply chain management is also an area concerning the revenue maximization problems. Lariviere and Porteus (2001) study a simple price-only contract where the manufacturer decides a wholesale price first and then the retailer decides an order quantity based on a random demand. The problem faced by the manufacturer is complex. Under IGFR, they show that the revenue function of the manufacturer is unimodal and then an optimal solution can be obtained analytically. We re-study the problem above and show that the manufacturer's problem can be solved analytically without IGFR.

We extend the transformation method to study a parametric cost minimization problem and get an equivalent one where the cost function is increasing, continuous and convex. A typical application of the cost minimization problem is the optimal control of service rate in queueing systems. As said in Stidham (2002), the Lippman device (Lippman, 1975) opened the gates for the application of Markov decision processes theory to queueing control problems. The idea of the Lippman device is to transform the underlying Markov process into an equivalent one in which the times between transitions are exponential random variables with a constant parameter. By applying his device, Lippman (1975) studies the optimization problems in exponential queueing systems. Later, for the optimal control problem of arrivals, Helm and Waldmenn (1984) study a general framework with multi-server queues in a random environment. For the optimal control of service rate, Jo and Stidham (1983) study the optimization problems in $M/G/1$. Stidham and Weber (1989) consider the problem of controlling the service and/or arrival rates in queues, with the objectives of minimizing the total expected cost to reach state zero and average-cost minimization over an infinite horizon. They prove that an optimal policy is monotonic in the number of customers in the system. See the details in survey papers (Stidham, 1985, 2002). However, in the literature, the analytical tractability of the optimization problems is not concerned, though is very important in computing optimal policies. Applying the transformation method, we solve the analytical tractability for an optimal service rate control in queueing systems.

The rest of the paper is organized as follows. In Section 2, we present the model of a parametric revenue maximization problem and transform it into an equivalent well structured one with a regular revenue function. Then in Section 3, we apply the transformation to study a continuous time revenue management problem without assumptions on the demand function. In Section 4, we apply the transformation method to re-study a supply chain with price-only contract. In Section 5, we generalize the transformation method to study a cost minimization problem and apply it to an optimization problem in queueing systems. Section 6 is a concluding section.

2. Parametric revenue maximization problem

In this section, we first present the model of the parametric revenue maximization problem. Without the usual conditions presented in the literature, we transform it into an equivalent one where the revenue function is increasing, continuous and concave.

2.1. Model

The model is based on a fairly standard price–demand formulation for a product (or service). There is a known mathematical relationship between price and demand. We let x denote price and y denote demand (demand in one time period, or per unit of time).

In the model, there is a parameter t from a nonempty set $\mathcal{T}$. t may represent status for decision epoch. We require no structure for $\mathcal{T}$ and so t may be multiple representing parameters. Suppose that for each t, price x is chosen from a nonempty set P and correspondingly a nonnegative demand $d(x)$ is received (called the demand function). It is initially assumed that P is a bounded set. This assumption will be relaxed in Remark 4 below. Then, a revenue $d(x)x$ is received. Furthermore, there is a cost $d(x)\lambda(t)$ after realizing the demand $d(x)$ at t. Here, $\lambda(t)$ is nonnegative and can be interpreted as unit opportunity cost for choosing x at t. Hence, we get a profit $d(x)x-d(x)\lambda(t)$ (called as revenue function) if x is choosing at t. We thus study the following parametric maximization problem:

$$\sup_{x\in P}\{d(x)x-d(x)\lambda(t)\},\quad t\in\mathcal{T}. \tag{1}$$

Note that this is, in fact, a family of maximization problems. We will give an example of $\lambda(t)$ in revenue management later. We want to get an optimal solution x_t^* for problem (1) for each $t\in\mathcal{T}$. For convenience, we say that x_t^* is optimal for problem $(1)_t$, or simply optimal for problem (1) when no confusion is induced.

2.2. Transformation

We study the maximization problem (1) according to the following steps. First, we reduce the price set P such that $d(x)$ is a one-to-one correspondence between price x and demand y: $y=d(x)$ and $x=p(y)$ for some function $p(y)$. So, we can transform the decision variable from price x into demand $y=d(x)$. Then, we reduce the domain of the revenue function $r(y):=yp(y)$ such that it is increasing. Finally, we revise $r(y)$ to be concave.

Step I: *An equivalent one with demand variable*. Denote by $\Lambda\equiv\{d(x)|x\in P\}$ the set of demands that are allowable under some price in P. For any given demand $y\in\Lambda$, denote by $P(y)\equiv\{x\in P|d(x)=y\}$ the set of prices that yield demand y. Surely, the set $P(y)$ may include multiple prices. But it may suffice to consider the largest one $p(y):=\sup P(y)$. The following lemma says that $p(y)$ is enough for the maximization problem (1) in the set $P(y)$. We denote by $b(x,t)=d(x)x-d(x)\lambda(t)$ for convenience.

Lemma 1. *For any given $y\in\Lambda$, suppose $p(y)\in P(y)$. Then, $b(p(y),t)\geq b(x,t)$ for all $x\in P(y)$ and $t\in\mathcal{T}$, where the equality holds if and only if $y=0$ or $x=p(y)$.*

Proof. For any given $y \in \Lambda$, suppose $p(y) \in P(y)$. For any $x \in P(y)$, because of $d(x) = y$ and $d(p(y)) = y$, we have $b(x,t) = d(x)[x - \lambda(t)] = d(p(y))[x - \lambda(t)]$. Due to the maximum of $p(y)$, we have further $b(x,t) \le d(p(y))[p(y) - \lambda(t)] = b(p(y),t)$ for $t \in \mathcal{T}$. Certainly, the inequality above becomes an equality if and only if $y = 0$ or $x = p(y)$. □

The condition $p(y) \in P(y)$ given in the lemma above is reasonable in practice. In fact, this condition is very weak, as shown in the following remark.

Remark 1. We have $p(y) \in P(y)$ under any one of the following conditions.

(a) $P(y)$ is a finite set. This is true in many practical cases, especially when P is finite.
(b) $d(x)$ is continuous in a closed price set P. In this case, $P(y)$ is also a closed set and so $p(y) \in P(y)$ for each $y \in \Lambda$.
(c) $d(x)$ is decreasing (in a nonstrict meaning), left continuous, bounded, and $d(\infty) := \lim_{x \to \infty} d(x) = 0$. In this case, $P(y)$ is also closed for each y. Let $D = d(0)$. Then, $d(x)$ can be expressed by

$$d(x) = D(1 - F(x)) \quad \text{for } x \ge 0 \tag{2}$$

where $F(x) = P\{\xi < x\}$ is a cumulative distributed function (d.f.) for some random variable ξ. In this case, D may be interpreted as the potential number of customers; if price is set as x, each potential customer buys one item of the product with probability $1 - F(x)$. Hence, $d(x)$ is the expected demand given price x, and $F(\cdot)$ is a cumulative distribution function representing customers' willingness-to-pay. Hence, $p(y) \in P(y)$ for each y. In Ziya et al. (2004), $d(x)$, and so $F(x)$, is further required to be twice differential. □

Following the remark above, we assume $p(y) \in P(y)$ for each $y \in \Lambda$ throughout this section. Due to Lemma 1, we take the price set as $P_1 = \{p(y) | y \in \Lambda\}$, a subset of the original set P. That is, the maximization problem (1) is equivalent to the following one:

$$\sup_{x \in P_1} \{d(x)[x - \lambda(t)]\}, \quad t \in \mathcal{T}.$$

Now, there is a one-to-one correspondence between prices in P_1 and demands in Λ, and $p(y)$ is the reverse function of $d(x)$. So, we can alternatively take demand y as the decision variable. Define the revenue as a function of demand by $r(y) = yp(y)$. Hence, the maximization problem (1) is equivalent to the following one:

$$\sup_{y \in \Lambda} \{r(y) - y\lambda(t)\}, \quad t \in \mathcal{T}. \tag{3}$$

That is, $x \in P$ is an optimal solution of $(1)_t$ if and only if $y = d(x)$ is an optimal solution of $(3)_t$ for each $t \in \mathcal{T}$. Here, the decision variable is demand y, instead of price x in (1). We denote by $B(y,t) = r(y) - y\lambda(t)$ for convenience. Surely, the function $B(y,t)$ is simpler than the original revenue function in (1). So, we deal with (3) in the following.

Though the maximization problem (3) is derived from the original problem (1), problem (3) itself may be an original one, e.g., in the optimization of queueing systems studied in Lippman (1975).

In the following we study the maximization problem (3) without condition on $r(y)$ and Λ.

Step II: *Increasingness of the revenue function*. First, we have the following lemma.

Lemma 2. *For any given demands $y_1, y_2 \in \Lambda$ with $y_1 < y_2$ and $r(y_1) > r(y_2)$, $B(y_1,t) > B(y_2,t)$ for all $t \in \mathcal{T}$. That is, such demand y_2 would not be optimal in problem* (3).

Proof. For any given $y_1, y_2 \in \Lambda$ with $y_1 < y_2$ and $r(y_1) > r(y_2)$, due to the nonnegative nature of $\lambda(t)$, $B(y_1,t) = r(y_1) - y_1\lambda(t) > r(y_2) - y_2\lambda(t) = B(y_2,t)$ for $t \in \mathcal{T}$. Hence, y_2 would not be optimal for each t. □

We call $y_2 \in \Lambda$ a *decreasing point* of $r(y)$ if there is $y_1 \in \Lambda$ such that $y_1 < y_2$ and $r(y_1) > r(y_2)$. Then, the lemma above tells us that all decreasing points in Λ are not optimal, and thus can be eliminated from the decision set Λ. We denote by Λ_1 the new decision set after the elimination, i.e.,

$$\Lambda_1 := \{y \in \Lambda | y \text{ is a nondecreasing point of } r(y)\}.$$

Surely, Λ_1 is a nonempty set and $r(y)$ is increasing in $y \in \Lambda_1$. When $r(y)$ is continuous in the closed set Λ, $\sup \Lambda_1$ achieves the supremum of $r(y)$ in $y \in \Lambda$, i.e., $r(\sup \Lambda_1) = \sup_{y \in \Lambda} r(y)$.

Due to Lemma 2, the maximization problem (3), and then (1), is equivalent to

$$\sup_{y \in \Lambda_1} \{r(y) - y\lambda(t)\}, \quad t \in \mathcal{T}. \tag{4}$$

In fact, $y \in \Lambda$ is optimal for problem (3) if and only if y is optimal for problem (4) and $y \in \Lambda_1$. For each $t \in \mathcal{T}$, let $y^*(t)$ be the largest maximizer in (4) if it exists. Then, we have the following result.

Proposition 1. (1) *The revenue function $r(y)$ is increasing, and so is continuous almost everywhere, in $y \in \Lambda_1$.* (2) *$y^*(t)$ is increasing (or decreasing) in t if $\lambda(t)$ is decreasing (or increasing) and $y^*(t)$ exists for each t.*

Proof. (1) This is true from real theory (see, e.g., Zheng and Wang, 1980). (2) This follows from the well-known property of modular functions (Topkis, 1998). □

$\lambda(t)$ is increasing in many maximization problems, e.g., in the revenue management, as discussed later. $y^*(t)$ exists if $r(y)$ is continuous in a closed set Λ_1. Part (2) of the proposition above tells us that when $\lambda(t)$ is monotone, the monotone of the optimal solution is robust to the revenue function $r(y)$. That is, whatever the revenue function (or the demand function in (1)) is, the optimal solution $y^*(t)$ to problem (3) increases (decreases) and so $p(y^*(t))$ is the optimal solution to problem (1) and is decreasing (increasing) when $\lambda(t)$ is decreasing (increasing).

Step III: *Concavity of the revenue function*. For problem (4), we further have the following lemma.

Lemma 3. *For any given y_1, y_2, $y_3 \in \Lambda_1$ with $y_1 < y_2 < y_3$, letting $\alpha = (y_3 - y_2)/(y_3 - y_1)$, if $r(y_2) < \alpha r(y_1) + (1-\alpha) r(y_3)$ then $B(y_2,t) < \max\{B(y_1,t), B(y_3,t)\}$ for all $t \in \mathcal{T}$. This means that y_2 would not be optimal for each $t \in \mathcal{T}$.*

Proof. For any given y_1, y_2, $y_3 \in \Lambda_1$ with $y_1 < y_2 < y_3$ and $\alpha = (y_3 - y_2)/(y_3 - y_1)$, $y_2 = \alpha y_1 + (1-\alpha) y_3$. So, when $r(y_2) < \alpha r(y_1) + (1-\alpha) r(y_3)$ we have

$$\begin{aligned} B(y_2,t) &= r(y_2) - y_2\lambda(t) \\ &< \alpha r(y_1) + (1-\alpha) r(y_3) - \alpha y_1 \lambda(t) - (1-\alpha) y_3 \lambda(t) \\ &= \alpha B(y_1,t) + (1-\alpha) B(y_3,t) \\ &\le \max\{B(y_1,t), B(y_3,t)\}, \quad t \in \mathcal{T}. \end{aligned}$$

Hence, y_2 would not be optimal at any $t \in \mathcal{T}$. □

We call point $y_2 \in \Lambda_1$ a *convex point* of $r(y)$ if there are $y_1, y_3 \in \Lambda_1$ satisfying the condition given in Lemma 3. Then, the lemma above tells us that any convex point of $r(y)$ would not be optimal for any t, and thus can be eliminated from the demand set Λ_1. Denote by Λ_2 the demand set after this elimination, i.e.,

$$\Lambda_2 := \{y \in \Lambda_1 | y \text{ is not a convex point of } r(y)\}.$$

According to Lemma 3, Λ_2 can be obtained as follows. For any two points $y_1, y_3 \in \Lambda_1$, delete all points in the curve $(y, r(y))$ which are below the line from point $(y_1, r(y_1))$ to point $(y_2, r(y_2))$ until no point can be deleted for arbitrary y_1 and y_3. Certainly, if $\inf \Lambda_1 \in \Lambda_1$ then $\inf\Lambda_1 \in \Lambda_2$, and also if $\sup \Lambda_1 \in \Lambda_1$ then $\sup \Lambda_1 \in \Lambda_2$. So, when Λ_1 is closed we have

$$\inf \Lambda_1 = \inf \Lambda_2, \quad \sup \Lambda_1 = \sup \Lambda_2. \tag{5}$$

Therefore, the maximization problem (4), and so problems (3) and (1), is equivalent to

$$\sup_{y \in \Lambda_2} \{r(y) - y\lambda(t)\}, \quad t \in \mathcal{T}. \tag{6}$$

Here, $r(y)$ is increasing and concave, and so continuous in $y \in \Lambda_2$ from Hu and Meng (2000). In this time, the objective function in the problem above is continuous and then it achieves the maximum if the domain Λ_2 is closed. We let $\overline{\Lambda}_2$ be the closure set of Λ_2. The value of $r(y)$ in $y \in \overline{\Lambda}_2 - \Lambda_2$ can be well defined since $r(y)$ is continuous in $y \in \Lambda_2$. In fact, for any $y_0 \in \overline{\Lambda}_2 - \Lambda_2$ with any sequence $\{y_n\}$ in Λ_2 satisfying $y_n \to y_0$, it is easy to get from the continuity of $r(y)$ that $r_0 := \lim_{n\to\infty} r(y_n)$ exists uniquely and is independent of the sequence $\{y_n\}$. So, we define $r(y_0) = r_0$. Therefore, we extend the revenue function $r(y)$ into the domain $\overline{\Lambda}_2$. Due to (5), when Λ_1 is closed we have also that $\inf \Lambda_1 = \inf \overline{\Lambda}_2$ and $\sup \Lambda_1 = \sup \overline{\Lambda}_2$. Then, we introduce the following problem with the same objective function as in (6) but an extended domain $\overline{\Lambda}_2$:

$$\sup_{y \in \overline{\Lambda}_2} \{r(y) - y\lambda(t)\}, \quad t \in \mathcal{T}. \tag{7}$$

From real theory (see Zheng and Wang, 1980), $\overline{\Lambda}_2 = \bigcup_i [a_i, b_i]$ with some constants $0 \le a_1 \le b_1 < \ a_2 \le b_2 \cdots \le d^*$, where $d^* = \sup \Lambda_2$. Let $c^* := \inf \Lambda$ be the lowest allowable demand. If $c^* \in \Lambda$ then c^* is surely neither a decreasing point nor a convex point, i.e., $c^* \in \Lambda_2$ and $a_1 = c^*$. Certainly, $\overline{\Lambda}_2 \subset [c^*, d^*]$ and $c^*, d^* \in \overline{\Lambda}_2$.

We further extend the revenue function $r(y)$ from the domain $y \in \overline{\Lambda}_2$ into the closed interval $y \in [c^*, d^*]$ as follows:

$$r^*(y) = \begin{cases} r(y) & a_i \le y \le b_i, \quad i = 1, 2, \ldots \\ \beta_i r(b_i) + (1-\beta_i) r(a_{i+1}) & b_i < y < a_{i+1}, \quad i = 1, 2, \ldots \end{cases} \tag{8}$$

where $\beta_i = (a_{i+1} - y)/(a_{i+1} - b_i)$. This extension is unique.

Due to Proposition 1 and Lemma 3, we have apparently the following proposition.

Proposition 2. *The revenue function $r^*(y)$ is increasing, continuous and concave in $y \in [c^*, d^*]$.*

The definition of the regular demand functions given in Gallego and van Ryzin (1994) does not require the increasingness. Based on the proposition above, we call a revenue function is regular if it is increasing, continuous and concave. Then, $r^*(y)$ is now regular.

With the regular revenue function $r^*(y)$, we consider the following maximization problem:

$$\sup_{y \in [c^*, d^*]} \{r^*(y) - y\lambda(t)\}, \quad t \in \mathcal{T}. \tag{9}$$

For each $t \in \mathcal{T}$, the objective function above is concave and continuous in the closed interval $[c^*, d^*]$. So, there are optimal solutions, the set of which may be an interval, denoted by $Y^*(t)$. Now, $Y^*(t)$ can be obtained by solving the first order condition of (9), i.e., $(d/dy) r^*(y) = \lambda(t)$ for $t \in \mathcal{T}$.

In some case, we can take $c^* = 0$ in the above proposition.

Remark 2. When for each $t \in \mathcal{T}$ there is $y \in [c^*, d^*]$ such that $r^*(y) - y\lambda(t) \ge 0$, we can add zero to $\overline{\Lambda}_2$ without changing the optimality of problem (6). Thus, we can take $c^* = 0$ and $r^*(y)$ can be extended to the domain $[0, d^*]$, as done in (8). □

2.3. Equivalent results

We have established the equivalence of the optimal solutions among problems (1), (3), (4), and (6). But we have not extend such an equivalence to problem (7) or (9). Overall, we have the following theorem.

Theorem 1. (1) *Any optimal solution of problem* (3), (4) *or* (6) *remains optimal for problems* (7) *and* (9).

(2) *For each $t \in \mathcal{T}$, $Y^*(t) \cap \overline{\Lambda}_2$ is the nonempty set of optimal solutions for problem* (7). *Moreover, when $Y^*(t) \cap \Lambda_2$ is nonempty, each of its elements is optimal for problems* (3), (4) *and* (6).

Proof. (1) It is obvious from the previous discussions.

(2) Given t, let $B^*(y,t) = r^*(y) - y\lambda(t)$. For any $y^0 \in [c^*, d^*] - \overline{\Lambda}_2$, there must exist $i \in \{1, 2, \ldots\}$ such that $b_i < y^0 < a_{i+1}$. Let $\beta = (a_{i+1} - y^0)/(a_{i+1} - b_i)$. Then, $y^0 = \beta b_i + (1-\beta) a_{i+1}$. Due to (8) and (9) we have that

$$\begin{aligned} B^*(y^0, t) &= \beta r(b_i) + (1-\beta) r(a_{i+1}) - y^0 \lambda(t) \\ &= \beta r(b_i) + (1-\beta) r(a_{i+1}) - \beta b_i \lambda(t) - (1-\beta) a_{i+1} \lambda(t) \\ &= \beta B^*(b_i, t) + (1-\beta) B^*(a_{i+1}, t) \\ &\le \max\{B^*(b_i, t), B^*(a_{i+1}, t)\} \end{aligned}$$

where the inequality follows due to $\beta \in (0,1)$. The inequality above becomes an equality if and only if $B^*(b_i, t) = B^*(a_{i+1}, t)$. Hence, both b_i and a_{i+1} are optimal when y^0 is optimal. This shows that $Y^*(t) \cap \overline{\Lambda}_2$ is nonempty. Then, the remaining result is clear from (1). □

The theorem above tells us two things. One is that any optimal solution of the original problem (3) remains optimal for problem (9). So, if there is an optimal solution of (3) then problem (9) must have optimal solutions. The other is that if $Y^*(t) \cap \Lambda_2$ is empty then the original problem (3) has no optimal solution; otherwise, each one in $Y^*(t) \cap \Lambda_2$ is optimal for problems (3), (4), (6) and (7).

In what follows, we consider a case where $\lambda(t) > 0$ for all $t \in \mathcal{T}$. First, we note that for this case, Lemma 2 can be revised as follows with exactly the same proof: for any $y_1, y_2 \in \Lambda$ with $y_1 < y_2$ and $r(y_1) \ge r(y_2)$, we have $B(y_1, t) > B(y_2, t)$ for all $t > 0$. We then redefine the decreasing points as such y_2. Thus, after eliminating all decreasing points, the revenue function $r(y)$ is strictly increasing in $y \in \Lambda_1$ and so does in $y \in \Lambda_2$ and in $y \in \overline{\Lambda}_2$.

With this revision, $r^*(y)$ is strictly increasing in $[c^*, d^*]$ and continuous and so bounded. Then, the maximization problem (9) has the unique optimal solution, denoted by $y^*(t)$, for each $t \in \mathcal{T}$. Therefore, we have the following corollary immediately from Theorem 1.

Corollary 1. *Suppose $\lambda(t) > 0$ for all $t \in \mathcal{T}$. Then, for each $t \in \mathcal{T}$ there exists the unique solution $y^*(t)$ of the maximization problem* $(9)_t$. *Moreover, $y^*(t)$ is the unique optimal solution of the maximization problem* $(3)_t$ *if and only if $y^*(t) \in \Lambda_2$.* □

If the original maximization problem is (1) and it has optimal solutions, for example, when $d(x)$ is continuous in the closed set P (this is often assumed in the literature). Then we have the following corollary.

Corollary 2. *Suppose $\lambda(t) > 0$ for all $t \in \mathcal{T}$ and problem* (1) *has optimal solutions. Then, for each $t \in \mathcal{T}$, $p(y^*(t))$ is the optimal solution of the maximization problem* $(1)_t$ *if and only if $y^*(t)$ is the optimal solution of the maximization problem* $(9)_t$. □

It should be noted that in the above steps, the revenue $r^*(y)$ and the interval $[c^*, d^*]$ are all independent of t.

2.4. Algorithms

In this subsection, we present an algorithm to compute Λ_1 and Λ_2 when $r(y)$ is continuously twice differentiable. In this case, the computation of Λ_1 and Λ_2 becomes simpler.

First, suppose $r(y)$ is continuously differentiable in a closed set Λ. Then, Λ_1 can be obtained by a simpler method as follows. In this case, $r'(y)$ is continuous in $y \in \Lambda$, and so $r(y)$ is increasing in some intervals and decreasing in other intervals. Thus, we can delete all those intervals in which $r(y)$ is decreasing. The remaining part of Λ consists of several intervals in which $r(y)$ is increasing. For these intervals, let $[a_1', b_1']$ be the first one. (Note: If $r(y)$ is decreasing at c^* then $a_1' = b_1' = c^*$.) Then, all points $a > b_1'$ satisfying $r(a) < r(b_1')$ should further be deleted. If the remaining part of Λ is nonempty, we denote by $[a_2', b_2']$ the first interval. Again all points $a > b_2'$ satisfying $r(a) < r(b_2')$ should be deleted, and so on. Finally, we have $\Lambda_1 = [a_1', b_1'] \cup [a_2', b_2'] \cup \cdots$, union of possibly finite or infinite intervals. Since demand is often bounded above, we assume for convenience $\Lambda_1 = [a_1', b_1'] \cup [a_2', b_2'] \cup \cdots \cup [a_n', b_n']$ with finite intervals. However, the infinite setting can be studied similarly and all results are still true in the following.

Suppose further $r(y)$ is continuously twice differentiable in $y \in \Lambda_1$. In this case, $r''(y)$ is continuous, and so $r'(y)$ is decreasing (or equivalently $r(y)$ is concave) in some intervals and is increasing (or $r(y)$ is convex) in other intervals. Similar to Λ_1, we let Λ_2' be the subset of Λ_1 in which $r'(y)$ is decreasing, i.e., $\Lambda_2' := \{y \in \Lambda_1 | r'(y)$ is decreasing at $y\} = [a''_1, b''_1] \cup [a''_2, b''_2] \cup \cdots \cup [a''_n, b''_n]$. Then, $r(y)$ is increasingly concave in each interval $[a''_i, b''_i]$, but not necessarily concave in Λ_2'.

It is obvious that $r^*(y)$ is concave in Λ_2' if and only if

$$r''(y) \le 0, \quad y \in \left[a_i'', b_i''\right]; \quad r'\left(b_i''\right) \ge \frac{r(a_{i+1}'') - r(b_i'')}{a_{i+1}'' - b_i''} \ge r'(a_{i+1}'') \quad \forall i = 1, 2, \ldots, n. \tag{10}$$

Since $r(y)$ is concave in each interval in Λ_2', we have the following algorithm.

Algorithm 1. Compute Λ_2 from $\Lambda_2' = [a''_1, b''_1] \cup [a''_2, b''_2] \cup \cdots \cup [a''_n, b''_n]$ for $n \ge 2$ provided that $r(y)$ is continuously twice differentiable.

Step 0: Let $i = 1$.

Step 1: Let $\delta = (r(a''_{i+1}) - r(b''_i))/(a''_{i+1} - b''_i)$.

Step 2: (**1**) If $r'(b''_i) < \delta$ then solve $(r(a''_{i+1}) - r(y))/(a''_{i+1} - y) = r'(y)$ in $[a''_i, b''_i)$. If there is a solution y_i^* of the equation in $[a''_i, b''_i)$, then let $b''_i = y_i^*$; otherwise, if $i = 1$ let $b''_i = a''_i$, else delete $[a''_i, b''_i]$ from Λ_2' and stop when $i = n-1$. Let $i = i+1$. Goto Step 1.

(**2**) If $r'(a''_{i+1}) > \delta$, then solve $(r(y) - r(b''_i))/(y - b''_i) = r'(y)$ in $[a''_{i+1}, b''_{i+1})$. If there is a solution y_{i+1}^* of the equation in $[a''_{i+1}, b''_{i+1})$, then let $a''_{i+1} = y_{i+1}^*$; otherwise, if $i = n$ let $a''_{i+1} := b''_{i+1}$ and stop, else delete $[a''_{i+1}, b''_{i+1}]$ from Λ_2' and stop when $i = n-1$. Let $i = i+1$. Goto Step 1.

It is clear that we get Λ_2 when Algorithm 1 stops.

We give the following remark about the concavity of $r(y)$.

Remark 3. (1) The complexity of Algorithm 1 mainly depends on solving the two equations in Step 2. If there is no analytic methods to solve these two equations, we need to solve them numerically. However, once $r^*(y)$ is obtained, the maximum of $r^*(y) - y\lambda(t)$ can be solved analytically for all $t \in \mathcal{T}$. Hence, this algorithm has advantage over the numerical computation when the parameter set $\mathcal{T}$ is large, e.g., an interval. On the other hand, it is easy to perform Algorithm 1 in many cases, as done in the example discussed in Section 3.3.

(2) It may be not necessary to compute $\overline{\Lambda}_2$ and $r^*(y)$. In fact, we can stop our steps whenever we can solve optimal solutions of problem (3), (4), (6), (7) and (9), or

$$\sup_{y \in \Lambda_2'} \{r(y) - y\lambda(t)\}, \quad t \in \mathcal{T}. \tag{11}$$

If we can get an optimal solution for any problem above then we no longer compute $\overline{\Lambda}_2$ to solve the problem (9). □

Finally, we give the following remark for the unboundedness of the price set P.

Remark 4. Suppose the price set P is unbounded. It is further assumed that $\lim_{x\to\infty} x d(x) = 0$, as usual in the literature, e.g., Gallego and van Ryzin (1994). Thus, $\lim_{x\to\infty} d(x) = 0$ and so $c^* = \inf \Lambda = 0$. We then define $r(0) := \lim_{y\to 0} r(y) = \lim_{x\to\infty} x d(x) = 0$. Furthermore, the objective in problem (1) is positive when $x > \lambda(t)$. This implies that the objective in problem (3) is positive at some y. Therefore, $y = 0$ would not be optimal for problem (3), and so we can assume that $0 \in \Lambda$ in (3). Then, all results in this section are true. □

2.5. Optimal arrival control in queueing systems

As an example to illustrate our problem and results discussed in the previous subsections, we restudy the optimal arrival control in a M/M/K system that is studied in Lippman (1975). This problem is not concerned in Ziya et al. (2004). Here, (1) customers arrive according to a Poisson process with rate λ, which is chosen from a nonempty compact set $\Lambda \subset [0, \overline{\lambda}]$ with $\overline{\lambda}$ being a finite and positive constant; (2) each of K servers serves customers with an exponential distributed time with rate μ. All the service times are independent with each other and also with the arrival process. The holding cost rate $h(i)$ when the queue length is i (i.e., i customers in the system) is nonnegative, increasing and convex. On the other hand, the system incurs a nonnegative reward rate $q(\lambda)$ when $\lambda \in \Lambda$ is chosen.

Let $V_\alpha^a(i)$ be the minimal discounted cost in an infinite horizon with a discount factor $\alpha > 0$. Then applying the Lippman's device, $V_\alpha^a(i)$ satisfies the following optimality equation:

$$V_\alpha^a(i) = \frac{1}{\Delta + \alpha}\{h(i) + \mu(i \wedge K)V_\alpha^a(i-1) + [\Delta - \mu(i \wedge K)]V_\alpha^a(i)\} + \frac{1}{\Delta + \alpha}\min_{\lambda \in \Lambda} g_\alpha^a(i, \lambda), \quad i \ge 0, \tag{12}$$

where $\Delta = \overline{\lambda} + \mu K$, $g_\alpha^a(i, \lambda) = -\lambda q(\lambda) + \lambda v_\alpha^a(i+1)$ and $v_\alpha^a(i+1) = V_\alpha^a(i+1) - V_\alpha^a(i)$. Define $\lambda_\alpha(i)$ as the largest minimizer in the optimality equation above. Lippman (1975) shows that if $q(\lambda)$ is either continuous or decreasing and right-continuous, then $V_\alpha^a(i)$ is convex in i and so $\lambda_\alpha(i)$ is decreasing in i. In Lippman (1975), the right continuity of $q(\lambda)$ ensures the existence of $\lambda_\alpha(i)$ and the decreasingness of $q(\lambda)$ ensures the monotone of $\lambda_\alpha(i)$. However, both the existence and monotone of $\lambda_\alpha(i)$ are ensured when $q(\lambda)$ is continuous. Furthermore, it is not concerned how to compute $\lambda_\alpha(i)$ in the literature.

Since $\min_\lambda g_\alpha^a(i, \lambda) = -\max_\lambda\{\lambda q(\lambda) - \lambda v_\alpha^a(i+1)\}$, the minimization problem $\min_\lambda g_\alpha(i, \lambda)$ in (12) can be fit into the problem (3). Given any function $q(\lambda)$, let Λ_1 be the subset of Λ consisting of nondecreasing points of $\lambda q(\lambda)$ and Λ_2 be the subset of Λ_1 consisting of concave points of $\lambda q(\lambda)$. Since Λ is compact, due to Theorem 1,

$$\min_{\lambda \in \Lambda} g_\alpha^a(i, \lambda) = \min_{\lambda \in \Lambda_1} g_\alpha^a(i, \lambda) = \min_{\lambda \in \Lambda_2} g_\alpha^a(i, \lambda), \quad i \ge 1. \tag{13}$$

So, the following proposition is clear from Proposition 1 and Theorem 1.

Proposition 3. *$V_\alpha^a(i)$ is convex in i, $\lambda_\alpha(i)$ is a solution of $(\lambda q(\lambda))' = v_\alpha^a(i+1)$ in Λ_2 and is decreasing in i.*

Only the monotone of the optimal policies is studied in the literature of optimization in queueing systems. We show this is true without any condition on $q(\lambda)$. Moreover, it is not concerned how to compute the optimal arrival rate $\lambda_\alpha(i)$, which is now the solution of $(\lambda q(\lambda))' = v_\alpha^a(i+1)$ in Λ_2.

3. Revenue management with dynamic pricing

In this section, we apply the transformation presented in Section 2 to study a revenue management with continuous time dynamic pricing.

3.1. Model

The problem we considered here is formulated as follows. The retailer has a stock of N items and wants to sell them during a finite time horizon T. At any time, the retailer chooses one price from an allowable price set P, which is arbitrary but nonempty. Demand arrives according to a nonhomogeneous Poisson process with a time-dependent demand intensity (or demand rate), denoted by $d_t(x)$, a function of price x. We only assume that $d_t(x)$ satisfies the condition given in Lemma 1. (This problem with the homogeneous demand is first studied by Gallego and van Ryzin, 1994).

Let N_t be the number of items sold out up to time t. A pricing policy is defined as a function $\mu = (x_t, 0 \le t \le T)$, with $x_t \in P$ for all t, satisfying $\int_0^T dN_t \le N$. This inequality means that the total number of items sold out must be less than or equal to N, the initial number hold in the retailer. The set of all pricing policies is denoted by U. For any given $\mu \in U$, the expected total revenue of the retailer by using pricing policy μ in the time period $[0,t]$ with the initial stock number n is

$$J_\mu(n,t) = E_\mu\left\{\int_0^t x_s dN_s\right\}, \quad 0 \le t \le T, \ \ n = 0, \ 1, \ldots, N.$$

Surely, we have the boundary conditions $J_\mu(n,0) = 0$ and $J_\mu(0,t) = 0$, which mean that no value remains if there remains no items or no time for selling.

The retailer's problem is to find a pricing policy that maximizes the expected total revenue over the policy set U: $J(n,t) = \sup_{\mu \in U} J_\mu(n,t)$. Here, $J(n,t)$ is called the optimal value function. It is well-known that $J(n,t)$ satisfies the following equation (the Hamilton–Jacobi Bellman (HJB) equation in Gihman and Skorohod, 1979, or the optimality equation in continuous time Markov decision process in Hu, 1993):

$$\frac{\partial J(n,t)}{\partial t} = \sup_{x \in P}\{d_t(x)[x - \Delta J(n,t)]\}, \quad t \in \mathcal{T}, \ \ n = 0,1,2,\ldots,N \tag{14}$$

where $\Delta J(n,t) = J(n,t) - J(n-1,t)$ is the marginal revenue at time t with n items. Let $x^*(n,t)$ be the largest maximizer in (14). Then, $x^*(n,t)$ is an optimal price of the retailer at (n,t).

In the following subsection, we study the problem (14) above for the homogeneous demand case and the nonhomogeneous demand case, respectively, by applying the transformation presented in Section 2.

3.2. Results

Since the demand function $d_t(x)$ depends on t, we know from Section 2 that for each t there are constants $0 \le c_t^* \le d_t^*$ and a regular revenue rate function $r_t^*(y)$ such that $J(n,t)$ satisfies the HJB equation (14) if and only if it satisfies the following HJB equation:

$$\frac{\partial J(n,t)}{\partial t} = \sup_{y \in [c_t^*, d_t^*]}\{r_t^*(y) - y\Delta J(n,t)\}, \quad t \in [0,T], \ \ n = 1,2,\ldots,N, \tag{15}$$

where $r_t^*(y)$ is strictly increasing and concave in y, and the optimal demand rate $y^*(n,t)$ is the unique maximizer in (15). The unique difference between Eqs. (15) and (14) is that the revenue and decision sets in Eq. (15) are dependent of the time variable t. We take price x instead of demand rate y as the decision variable. Then, Eq. (15) is equivalent to

$$\frac{\partial J(n,t)}{\partial t} = \sup_{x \in [p_{t1}, p_{t2}]}\{d_t^*(x)[x - \Delta J(n,t)]\}, \quad t \in [0,T], \ \ n = 1,2,\ldots,N$$

where $d_t^*(x)$ is the demand function, defined as the reverse of the function $r_t^*(y)/y$, and $p_{t1} = r_t^*(d_t^*)/d_t^*$ and $p_{t2} = r_t^*(c_t^*)/c_t^*$. Apparently, $J(n,t)$ is increasing in t and so $\partial J(n,t)/\partial t \ge 0$. Thus, there must be $x \in [p_{t1}, p_{t2}]$ such that $d_t^*(x)[x - \Delta J(n,t)] \ge 0$. Therefore, the HJB equation above is further equivalent to

$$\frac{\partial J(n,t)}{\partial t} = \sup_{x \ge 0}\{d_t^*(x)[x - \Delta J(n,t)]\}, \quad t \in [0,T], \ \ n = 1,2,\ldots,N \tag{16}$$

where we define $d_t^*(x) = 0$ if $x \notin [p_{t1}, p_{t2}]$. In the equation above, the price is constrained only to be nonnegative. Then, the optimal price $x^*(n,t)$ is the unique maximizer in (16), due to the unique maximizer of (15). It should be noted that Eq. (16) has the same form as the original HJB equation (14). But, here the price set is $[0,\infty)$ and $d_t^*(x)$ is regular due to Proposition 2. Obviously, Eq. (16) is exactly equation (4) in Wei and Hu (2002). Hence, we have the following results from Theorems 1, 5, and 6 in Wei and Hu (2002).

Theorem 2. *For any given t, the optimal expected revenue function $J(n,t)$ is increasing and concave in n, the optimal pricing policy $x^*(n,t)$ is decreasing in n, and the optimal demand rate $y^*(n,t)$ is increasing in n. Moreover, $x^*(n,t)$ is increasing in t when $d_t^*(x_1)/d_t^*(x_2)$ is increasing in t for each given $x_1 > x_2$.*

The theorem above implies that the more the remaining items are, the lower the price will be, and under the given condition on $d_t^*(x)$ in the theorem, the longer the selling horizon is, the higher the price will be.

It is interesting that we used the transformation method twice above: first to derive Eq. (15) from (14) and second to derive Eq. (16) from (15).

When the demand function is homogeneous, i.e., $d_t(x) = d(x)$ is irrespective of t, all c_t^*, d_t^*, $r_t^*(y)$ in (15) are homogeneous, denoted by $c^*, d^*, r^*(y)$, respectively. In fact, $c^* = \inf_{x \in P} d(x)$ and $d^* = \arg\sup yp(y)$. So, HJB equation (15) is exactly equation (8) in Gallego and van Ryzin (1994). Thus, the following corollary follows from Theorem 1 there.

Corollary 3. *When the demand is homogeneous, $J(n,t)$ is increasing and concave in t, both $\Delta J(n,t)$ and $x^*(n,t)$ are increasing in t, but $y^*(n,t)$ is decreasing in t.*

In Theorem 2 and Corollary 3, we show monotone properties of the optimal demand rate and the optimal price. These are shown in the literature under the condition that the demand function is regular (Gallego and van Ryzin, 1994). Our results imply that these monotone properties are robust to the demand function (together with the price set). This is true, in fact, irrespective of the regularity of the demand function, as shown in Proposition 1. The regularity is to ensure that the optimal solution can be obtained from the first order condition.

In Section 2, the demand function $d(x)$ and the revenue function $r(y)$ are independent of the parameter. Here, we generalize the transformation to the case where $d_t(x)$ and $r_t(y)$ depend on the parameter t.

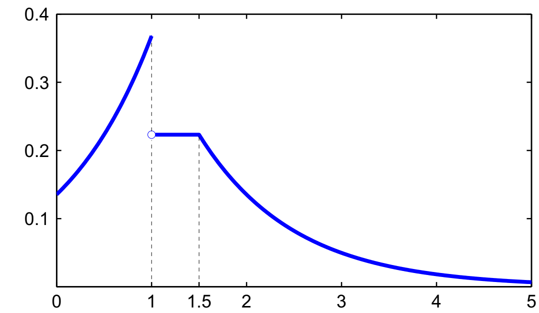

Fig. 1. The demand function $d(x)$.

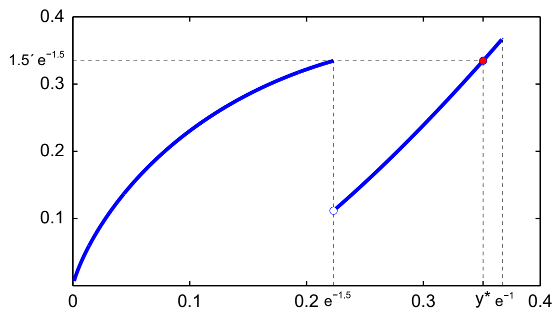

Fig. 2. The revenue function $r(y)$.

3.3. An example

Consider a homogeneous revenue management problem where the price set is $P=[0,\infty)$ and the demand rate function is

$$d(x)=\begin{cases}\min\{e^{x-2},e^{-x}\}, & x\le 1,\ x>1.5\\ e^{-1.5}, & 1<x\le 1.5.\end{cases}$$

Here, $d(x)$ is neither increasing nor decreasing (see Fig. 1). In fact, $d(x)=e^{x-2}$ is increasing in $x\le 1$; $d(x)=e^{-1.5}$ remains a constant in $x\in(1,1.5]$; while $d(x)=e^{-x}$ is decreasing in $x>1.5$. We apply the transformation for this problem in the following.

The value region of $d(x)$ is $\Lambda=(0,e^{-1}]$. For $y\in(0,e^{-1.7}),P(y)=\{-\ln y\}$ and so $p(y)=-\ln y$; for $y\in[e^{-1.7},e\ \ -1.5),P(y)=\{-\ln y,2+\ln y\}$ and so $p(y)=-\ln y$; for $y=e^{-1.5},P(y)=(1,1.5]$ and so $p(y)=1.5=-\ln y$; while for $y\in(e^{-1.5},e^{-1}],P(y)=\{2+\ln y\}$ and so $p(y)=2+\ln y$. Thus, the revenue function with y is given by

$$r(y)=\begin{cases}-y\ln y, & y\in(0,e^{-1.5}]\\ y(2+\ln y), & y\in(e^{-1.5},e^{-1}].\end{cases}$$

See $r(y)$ in Fig. 2.

Therefore, the HJB equation (14) for the problem here is equivalent to the following one:

$$\frac{\partial J(n,t)}{\partial t}=\sup_{y\in(0,e^{-1}]}\{r(y)-y\Delta J(n,t)\},\quad t\in[0,T],\ \ n=1,2,\ldots,N.$$

Moreover, since $\lim_{y\to0+}y\ln y=0$, $r(y)$ is strictly increasing in $y\in(0,e^{-1.5}]$ and in $y\in(e^{-1.5},e^{-1}]$, but decreasing at $y=e^{-1.5}$. Let y^* be the unique solution of equation $r(y)=r(e^{-1.5}-)=1.5e^{-1.5}$ in $y\in(e^{-1.5},e^{-1}]$, i.e., $y^*(2+\ln y^*)=1.5e^{-1.5}$. Then $y^*\approx0.3511<e^{-1}$. Therefore, $\Lambda_1=(0,e^{-1.5}]\cup(y^*,e^{-1}]$. Obviously, $r(y)$ is concave in $y\in(0,e^{-1.5}]$ but convex in $y\in(y^*,e^{-1}]$. Thus, $\Lambda_2=(0,e^{-1.5}]\cup\{e^{-1}\}$. So, we can limit us to consider the optimal solution for the HJB equation in the set Λ_2. That is, it suffices to consider the HJB equation

$$\frac{\partial J(n,t)}{\partial t}=\sup_{y\in(0,e^{-1.5}]\cup\{e^{-1}\}}\{r(y)-y\Delta J(n,t)\},\quad t\in[0,T],\ \ n=1,2,\ldots,N.$$

We do not need to compute $r^*(y)$ and c^*,d^* here. It is easy to see that the optimal solution for the HJB equation above is

$$y^*(n,t)=\begin{cases}e^{-1} & \text{if } \Delta J(n,t)\le(e^{0.5}-1.5)/(e^{0.5}-1)\\ e^{-1.5} & \text{if } (e^{0.5}-1.5)/(e^{0.5}-1)\le\Delta J(n,t)\le0.5\\ e^{-1-\Delta J(n,t)} & \text{if } \Delta J(n,t)\ge0.5.\end{cases}$$

Thus, the optimal price is

$$x^*(n,t)=p(y^*(n,t))=\begin{cases}1 & \text{if } \Delta J(n,t)\le(e^{0.5}-1.5)/(e^{0.5}-1)\\ 1.5 & \text{if } (e^{0.5}-1.5)/(e^{0.5}-1)\le\Delta J(n,t)\le0.5\\ 1+\Delta J(n,t) & \text{if } \Delta J(n,t)\ge0.5.\end{cases}$$

So, for $t\in[0,T],\ n=1,2,\ldots,N$,

$$\frac{\partial J(n,t)}{\partial t}=\begin{cases}e^{-1}[1-\Delta J(n,t)] & \text{if } \Delta J(n,t)\le(e^{0.5}-1.5)/(e^{0.5}-1)\\ e^{-1.5}[1.5-\Delta J(n,t)] & \text{if } (e^{0.5}-1.5)/(e^{0.5}-1)\le\Delta J(n,t)\le0.5\\ e^{-1-\Delta J(n,t)} & \text{if } \Delta J(n,t)\ge0.5.\end{cases}$$

Since $\Delta J(n,t)$ is increasing in t, for each n there exist $t^*_{n1}<t^*_{n2}$ such that $\Delta J(n,t^*_{n1})=(e^{0.5}-1.5)/(e^{0.5}-1)$ and $\Delta J(n,t^*_{n2})=0.5$. So,

$$\frac{\partial J(n,t)}{\partial t}=\begin{cases}e^{-1}[1-\Delta J(n,t)] & \text{if } t\le t^*_{n1}\\ e^{-1.5}[1.5-\Delta J(n,t)] & \text{if } t^*_{n1}\le t\le t^*_{n2}\\ e^{-1-\Delta J(n,t)} & \text{if } t\ge t^*_{n2}.\end{cases}$$

Due to $J(0,t)=0$, the above differential equation can be solved iteratively for $n=1,2,\ldots,N$.

For example, for $n=1$, $J(0,t)=0,J(1,t)=\Delta J(1,t)$. Solving the differential equation $\partial J(1,t)/\partial t=e^{-1}[1-J(1,t)]$ with the boundary condition $J(1,0)=0$ we get[1] $J(1,t)=1-e^{-e^{-1}t}$. Solving $J(1,t)=(e^{0.5}-1.5)/(e^{0.5}-1)$ we get $t^*_{11}=-e\ln(2(e^{0.5}-1))\approx0.708$.

Then, solving the differential equation $\partial J(1,t)/\partial t=1.5e^{-1.5}-e^{-1.5}J(1,t),t\ge t^*_{11}$ with the boundary condition $J(1,t^*_{11})=(e^{0.5}-1.5)/(e^{0.5}-1)$ we get

$$J(1,t)=J(1,t^*_{11})e^{-e^{-1.5}(t-t^*_{11})}+\int_{t^*_1}^{t}e^{-e^{-1.5}(t-s)}1.5e^{-1.5}\,ds$$
$$=[J(1,t^*_{11})-1.5]e^{-e^{-1.5}(t-t^*_{11})}+1.5.$$

Solving $J(1,t)=0.5$ for $t>t^*_{11}$ we get $t^*_{12}=t^*_{11}-e^{1.5}\ln(2(1-e^{-0.5}))\approx1.782$.

Finally, solving the differential equation $\partial J(1,t)/\partial t=e^{-1-J(1,t)},t\ge t^*_{12}$ with the boundary condition $J(1,t^*_{12})\ =0.5$ we get $J(1,t)=\ln[e^{-1}(t-t^*_{12})+e^{0.5}]$ for $t\ge t^*_{12}$. Therefore, we have

$$J(1,t)=\begin{cases}1-e^{-e^{-1}t} & \text{if } t\le t^*_{11}\\ (J(1,t^*_{11})-1.5)e^{-e^{-1.5}(t-t^*_{11})}+1.5 & \text{if } t^*_{11}\le t\le t^*_{12}\\ \ln[e^{-1}(t-t^*_{12})+e^{0.5}] & \text{if } t\ge t^*_{12}.\end{cases}$$

Figs. 3 and 4 give the optimal price $x^*(n,t)$ and the maximum expected total revenue $J(n,t)$ with $t=10,20,30$ and $n=1,2,\ldots,10$, respectively. The results perfectly illustrate our conclusions as shown in Theorem 2 and Corollary 3.

The computation for the example above shows that it is easy to implement our transformation. On the contrary, it is difficult to solve directly the revenue maximization problem (14) even for $d_t(x)=d(x)$.

4. A supply chain with price-only contract

Assumption IGFR is also used in Lariviere and Porteus (2001) for a supply chain with price-only contract for one period. The supply chain consists of one manufacturer and one retailer. The manufacturer, as the game's leader, produces the product with a

[1] The solution for the differential equation $f'(t)=a(t)f(t)+b(t)$ with the boundary condition $f(t_0)=y_0$ is $f(t)=y_0e^{\int_{t_0}^{t}a(x)\,dx}+\int_{t_0}^{t}e^{\int_s^t a(x)\,dx}b(s)\,ds,t\ge t_0$.

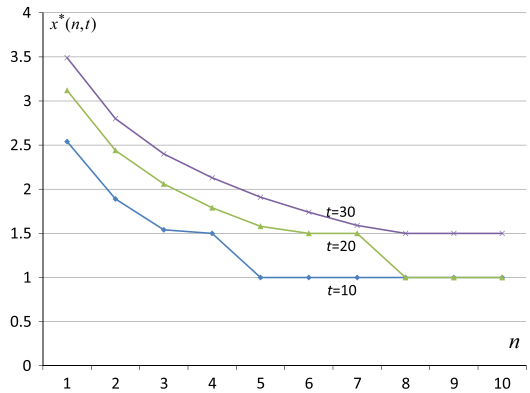

Fig. 3. The optimal price $x^*(n,t)$.

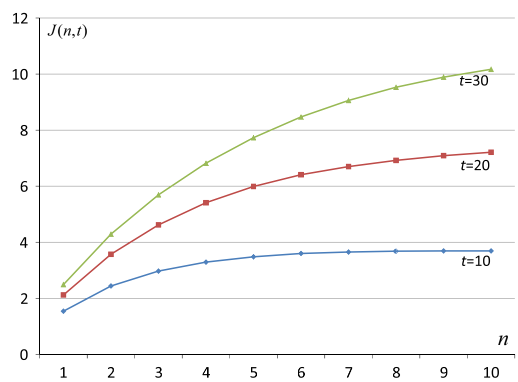

Fig. 4. The maximum revenue $J(n,t)$.

unit cost c and determines first the wholesale price w to sell her products to the retailer. The retailer, as the follower, determines his order quantity q, or equivalently the inventory level y after the order, based on the wholesale price w. The demand D at the retailer is a random variable with d.f. $F(\cdot)$. The retail price $r>c$ is exogenous. It is further assumed that the salvage value of unsold products at the end of the period is zero and unmet demands are lost. Let $F^{-1}(\cdot)$ be the reverse function of $F(\cdot)$ and $\overline{F}(y)=1-F(y)$.

For any given wholesale price $w \le r$, the retailer faces a newsvendor problem with the expected profit

$$\Pi_R(y) = -wy + rE\min\{y,D\} = -wy + r\int_0^y x\, dF(x) + r\overline{F}(y). \tag{17}$$

It is well-known that the retailer's optimal inventory level is $y(w)=F^{-1}((r-w)/r)$ given w.

Therefore, the manufacturer's demand is $y(w)$ and his profit is $\Pi_M(w)=(w-c)y(w)$, whenever his wholesale price is w. So, his problem is

$$\max_{c \le w \le r} \Pi_M(w) = (w-c)y(w). \tag{18}$$

Since $y(w)$ has the reverse function $w(y):=r\overline{F}(y)$, we can take y as the decision variable, instead of w, for the manufacturer. Therefore, the problem (18) becomes equivalently

$$\max_{y \ge 0} \Pi_M(y) = R(y) - cy, \tag{19}$$

where $R(y)=ry\overline{F}(y)$ is the revenue function when the retailer's inventory level is y.

Lariviere and Porteus (2001) shows that under Assumption IGFR (i.e., F is IGFR) and $F(\cdot)$ is derivative, $\Pi_M(y)$ is a unimodal function, and therefore the optimal solutions can be obtained easily. But IGFR is not needed if our transformation method is applied. By letting $\mathcal{T}=\{0\}$ be a singleton and $\lambda(t)=c$, we see that problem (19) is exactly problem (3). Then, from Corollary 1 we can get the following theorem. Here, $c^*=\inf \Lambda = 0$ due to $\Lambda=[0,\infty)$.

Theorem 3. *There is a revenue function $R^*(y)$, which is continuous, strictly increasing and concave, such that problem (19) is equivalent to the following one with $d^*=\arg\sup_{y\ge 0} R(y)$:*

$$\max_{y \in [0,d^*]} \{R^*(y) - cy\}. \tag{20}$$

Here, $R^*(y)-cy$ is continuous and concave. So, the above problem can be solved by solving its first order condition. Another advantage of problem (20) prior to problem (19) is that the domain here is a finite interval $[0,d^*]$.

Since there is no parameter t here, it is not necessary to construct $R^*(y)$. In fact, we can simplify the steps in Algorithm 1 for computing the optimal solution of (19).

Algorithm 2. Solving problem (19). **(1)** (Since the first order condition of (19) is $R'(y)=c$, its solutions must lie in Λ_1.) If $R'(y)=c$ has the unique solution y^* then it suffices to judge $R''(y^*)\le 0$. When $R''(y^*)\le 0$, y^* is the optimal solution of (19); otherwise (19) has no optimal solution.

(2) If $R'(y)=0$ has multiple solutions, we compute Λ_2. Due to Theorem 1, any solution of $R'(y)=0$ in Λ_2 is optimal for problem (19); otherwise if there is no solution of $R'(y)=0$ in Λ_2, then (19) has no optimal solution.

We consider two examples in the following to illustrate how to solve problem (19), for continuous type and discrete type random demands, respectively.

Example 1. The d.f. $F(y)$ of the demand and then the revenue function $R(y)$ are, respectively,

$$F(y)=\begin{cases} 0, & 0\le y\le 1, \\ \dfrac{1}{2}-\dfrac{1}{2y}, & 1\le y\le 2, \\ 1-\dfrac{3}{4(y-1)^2}, & y\ge 2, \end{cases} \qquad R(y)=ry\overline{F}(y)=\begin{cases} ry, & 0\le y\le 1, \\ \dfrac{r}{2}(y+1), & 1\le y\le 2, \\ \dfrac{3ry}{4(y-1)^2}, & y\ge 2. \end{cases}$$

Surely, $R(y)$ is increasing in $y\le 2$ and decreasing in $y\ge 2$. So, $\Lambda_1=[0,2]$. Moreover, $\overline{\Lambda}_2=\Lambda_2=\Lambda_1=[0,2]$. So, $R(y)-cy$ is concave in $y\in[0,2]$. Now, $(d/dy)[R(y)-cy]=r-c$ for $0\le y<1$ and $=\frac{1}{2}r-c$ for $1<y\le 2$. Due to $r>c$, we know that the optimal solution is $y=1$ when $c<r<2c$, is $y=2$ when $r\ge 2c$, and is any $y\in[1,2]$ when $r=2c$. □

The assumption of IGFR is not true here. In fact, its generalized failure rate is $e(y)=0$ for $0\le y<1$, $e(y)=1/y(y+1)$ for $1\le y<2$, and $e(y)=y/2(y-1)$ for $y\ge 2$. Surely, $e(y)$ is strictly decreasing in $y\in[1,2)$ and in $y>2$, but increasing only at $y=2$.

Hence, the example above illustrates that our method can be applied to cases where IGFR is not true and the generalized failure rate $e(x)$ may be strictly decreasing at the optimal solutions. Moreover, the IGFR needs that the d.f. is continuous type, but the method in this paper can unify the continuous and discrete types of d.f.s.

At the end of this section, we give the following example, which shows that the maximization problem (19) for the manufacturer may have no optimal solution for discrete type demand if the d.f. is defined by $F(x)=P\{\xi\le x\}$.

Example 2. Consider the demand variable with probability $P\{\xi=1\}=1/6, P\{\xi=2\}=1/2, P\{\xi=3\}=1/3$. The d.f. of the demand is defined by $F(y)=P\{\xi\le y\}$, which is right continuous.

Then, $F(y)$ and the revenue function $R(y)$ are, respectively,

$$F(y)=\begin{cases}0, & y<1,\\ \frac{1}{6}, & 1\le y<2,\\ \frac{2}{3}, & 2\le y<3,\\ 1, & y\ge 3,\end{cases}\qquad R(y)=ry\overline{F}(y)=\begin{cases}ry, & y<1\\ \frac{5}{6}ry, & 1\le y<2\\ \frac{1}{3}ry, & 2\le y<3\\ 0, & y\ge 3.\end{cases}$$

Then, $\Lambda_1=[0,1)\cup[\frac{6}{5},2)$, $\Lambda_2=[0,1)$, and $R^*(y)=ry\overline{F}(y)=ry$ for $y\in\overline{\Lambda}_2=[0,1]$. Obviously, the optimal solution of $\max_{y\in[0,1]}\{R^*(y)-cy\}$ is 1 when $r>c$. But due to Theorem 1, the original maximization problem (19) in this example has no optimal solution (in fact, the maximum value of the objective $R(y)-cy$ is achieved when $y\to 2-$ from the left side of point 2).

The example above indicates a fact that when $F(y)$ is increasing and right-continuous (a discrete type d.f.), problem (19) may have no optimal solution and the optimal solution of (20) may be not in Λ_1. This differs from the case when F is a continuous type d.f. for which the objective function is continuous and so there must have optimal solutions for (19).

5. Extension to a parametric cost minimization problem

In this section, we extend the transformation method presented in Section 2 to a parametric cost minimization problem. This is further illustrated by the optimal service rate control in a queueing system.

5.1. Parametric cost minimization problem

We consider the following problem:

$$\inf_{y\in\Lambda}\{r(y)-y\lambda(t)\},\quad t\in\mathcal{T},\tag{21}$$

where both $r(y)$ and $\lambda(t)$ are nonnegative, as in problem (3). Here, $r(y)$ represents cost for choosing y and $\lambda(t)$ represents revenue. So, we call the problem as *the cost minimization problem*. Later, we will discuss its application in the optimization of queueing systems. For this problem, we have the results similar to those in Section 2 with similar proof. In the following, we only give an outline. Let $C(y,t)=r(y)-y\lambda(t)$.

Lemma 4. *For any $y_1,y_2\in\Lambda$ with $y_1<y_2$ and $r(y_1)>r(y_2)$, $C(y_1,t)>C(y_2,t)$ for all $t\in\mathcal{T}$. Hence, such point y_1 would not be optimal in problem (21).*

From the lemma above, we delete all such points as y_1 and let

$$\Lambda':=\{y\in\Lambda|\text{there is no } y_2>y \text{ such that } r(y_2)<r(y)\}.$$

Surely, Λ' is a nonempty set and $r(y)$ is increasing in $y\in\Lambda'$. Then, the minimization problem (21) is equivalent to the following one:

$$\inf_{y\in\Lambda'}\{r(y)-y\lambda(t)\},\quad t\in\mathcal{T}.\tag{22}$$

Lemma 5. *For any given y_1, y_2, $y_3\in\Lambda'$ with $y_1<y_2<y_3$, letting $\alpha=(y_3-y_2)/(y_3-y_1)$, if $r(y_2)>\alpha r(y_1)+(1-\alpha)r(y_3)$ then $C(y_2,t)>\max\{C(y_1,t),C(y_3,t)\}$ for all $t\in\mathcal{T}$. This means that y_2 would not be optimal for each $t\in\mathcal{T}$.*

We call point y_2 satisfying the condition given in Lemma 5 as a *concave point* of $r(y)$. So, we can delete all concave points in Λ' and thus we let

$$\Lambda'':=\{y\in\Lambda'|y \text{ is not a concave point of } r(y)\}.$$

Therefore, the minimization problem (22), and so problems (21), is equivalent to

$$\inf_{y\in\Lambda''}\{r(y)-y\lambda(t)\},\quad t\in\mathcal{T}.\tag{23}$$

Here, $r(y)$ is increasing and convex, and so continuous in $y\in\Lambda''$ from Hu and Meng (2000).

Let $\overline{\Lambda}''$ be the closure set of Λ''. Then, we also consider the problem

$$\inf_{y\in\overline{\Lambda}''}\{r(y)-y\lambda(t)\},\quad t\in\mathcal{T}.\tag{24}$$

Let $d^*=\sup\Lambda''$ and $c^*:=\inf\Lambda''$. We extend the cost function $r(y)$ from the domain Λ'' into the domain $\overline{\Lambda}''$ in a natural way, and further into the closed interval $y\in[c^*,d^*]$ similar to that in (8). Now, $r^*(y)$ is increasing and convex in $[c^*,d^*]$, and we get the following minimization problem:

$$\inf_{y\in[c^*,d^*]}\{r^*(y)-y\lambda(t)\},\quad t\in\mathcal{T}.\tag{25}$$

Let $Y^*(t)$ be the set optimal solutions for the problem above. Therefore, we have the theorem similar to Theorem 1.

Theorem 4. (1) *Any optimal solution of problem (21), (22), or (23) remains optimal for problems (24) and (25).*

(2) *For each $t\in\mathcal{T}$, $Y^*(t)\cap\overline{\Lambda}''$ is the nonempty set of optimal solutions for problem (24); when $Y^*(t)\cap\Lambda''\neq\emptyset$, each of its elements is optimal for problems (21), (22) and (23).*

5.2. Optimal service rate control in queueing systems

As an example of the parametric cost minimization problem, we restudy the optimal service rate control in a M/M/1 system that is studied in Lippman (1975). Here, (1) customers arrive according to a Poisson process with rate λ; (2) the single server serves customers with an exponential distributed time with rate μ, which is chosen from a nonempty compact set $\Lambda\subset[0,\overline{\mu}]$ with $\overline{\mu}>0$. All the service times are independent with each other and also with the arrival process. The holding cost rate $h(i)$ is nonnegative, increasing and convex. Moreover, there is a nonnegative service cost rate $c(\mu)$ when $\mu\in\Lambda$ is chosen.

Let $V_\alpha^s(i)$ be the minimal discounted cost in an infinite horizon with a discount factor $\alpha>0$. Then applying Lippman's device, $V_\alpha^s(i)$ satisfies the following optimality equation:

$$V_\alpha^s(i)=\frac{1}{\Delta+\alpha}\left\{h(i)+\lambda V_\alpha^s(i+1)+\overline{\mu}V_\alpha^s(i)+\min_{\mu\in\Lambda}g_\alpha^s(i,\mu)\right\},\quad i\ge 0,\tag{26}$$

where $\Delta=\lambda+\overline{\mu}$, $g_\alpha^s(0,\mu)=0$, $g_\alpha^s(i,\mu)=c(\mu)-\mu[R+v_\alpha^s(i)]$ and $v_\alpha^s(i):=V_\alpha^s(i)-V_\alpha^s(i-1)$ for $i>0$. Define $\mu_\alpha(i)$ as the largest minimizer in the optimality equation above. Under the condition that $c(\mu)$ is either continuous or increasing and left-continuous and $h(i)=hi$ for some positive constant h, Lippman (1975) shows that $V_\alpha^s(i)$ is convex in i and so the optimal service rate $\mu_\alpha(i)$ is increasing in i. However, it is not concerned how to compute $\mu_\alpha(i)$ in the literature.

Differently from the optimal arrival control discussed in Section 2.5, the minimization problem $\min_{\mu\in\Lambda}g_\alpha^s(i,\mu)$ here is fit into the cost minimization problem (21). Thus, given any function $c(\mu)$, let Λ' be the subset of Λ and Λ'' be the subset of Λ' according to those in the last subsection. Since Λ is compact, both Λ' and Λ'' are also compact. So, from Theorem 4,

$$\min_{\mu\in\Lambda}g_\alpha^s(i,\mu)=\min_{\mu\in\Lambda'}g_\alpha^s(i,\mu)=\min_{\mu\in\Lambda''}g_\alpha^s(i,\mu),\quad i\ge 0.\tag{27}$$

Thus, we have the following proposition.

Proposition 4. *$V_\alpha^s(i)$ is convex in i, the optimal service rate $\mu_\alpha(i)$ is a solution of $c'(\mu)=R+v_\alpha^s(i)$ in Λ'' and is increasing in i.*

The optimization problems similar to $\min_{\lambda \in \Lambda} g_\alpha^a(i,\lambda)$ and $\min_{\mu \in \Lambda} g_\alpha^s(i,\mu)$ happen often in the optimal control in queueing systems, e.g., those studied in Stidham and Weber (1989), Jo and Stidham (1983) and Altman and Nain (1993). The transformation presented in this paper can be applied to these problems. So, the conditions presented in them can be ignored. Moreover, the computation of the optimal policies is not concerned in the literature of queueing systems.

6. Conclusions

In this paper, we present a parametric revenue maximization problem as a uniform framework for several problems studied in the literature. We transform the problem into an equivalent well structured one in which the revenue function is regular (increasing, continuous and concave), and so the problem becomes analytically tractable. Hence, we no longer need the usual assumptions presented in the literature. We apply this transformation method to study a continuous time revenue management without the usual assumptions and we get the usual results. An example is used to illustrate our method. Hence, we show the robustness of the monotone properties to the demand function. The transformation is also used to study a parametric cost minimization problem. We transform two optimal control problems of arrival rate and service rate in queueing systems to be analytically tractable, which has not been concerned in the literature. We also apply the transformation method to a supply chain with price-only contract.

Further research may include applying this transformation to other areas on revenue maximization problems, for example, in auctions where IGFR is used. Also, it may be interesting to relax assumptions and/or improve results for the other maximization problems in revenue management, e.g., those discussed in Sections 3–5 in Gallego and van Ryzin (1994), and in the optimal control of other queueing systems.

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